STABLE ∞-CATEGORIES

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1. Introduction

There is a useful comparison to be made between homotopy theory and homological algebra: for example, we talk about "contractible" chain complexes or "homotopies" between chain complexes. Stable ∞-categories can be viewed as a way to fit the derived ∞-category of an abelian category and the ∞-category of spectra into the same framework.

To take a more historical viewpoint, we have since the 1960s known one unification of the derived category and the homotopy category of spectra, namely as "triangulated categories". While these triangulated categories are very useful in practice, their theory lacks various desirable properties: for example, the category of functors between two triangulated categories doesn't inherit a triangulated structure, and it is often not possible to use "descent" (*i.e.* "gluing") arguments with them. Many of these problems can be traced to the fact that we are identifying things without remembering *why* they're identified. Using ∞-categories allows us to remember this extra information.

1.1. **Acknowledgements.** I would like to thank Lars Hesselholt, Kaif Hilman, and Robin Sroka for useful feedback on a previous draft of this document. I would also like to thank Malte Leip for a helpful discussion about structures and properties.

2. (Co)limits in quasicategories

We'll start with a very brief review of a particular model for ∞-categories, namely *quasicategories.* Our goal is to be able to talk about limits and colimits in this model. All the information in this section can be found in the first chapter of [HTT].

2.1. *Remark.* A brief remark on terminology: loosely speaking, an $(\infty, 1)$-category is a category that has n-morphisms for every $n \in \mathbb{Z}_{\geq 1}$ such that for $n > 1$, all n-morphisms are invertible. This is a bit cumbersome to write, so here we will by convention refer to these objects simply as "∞-categories".

There are many ways of making the above definition precise, and all such models that I am aware of are known to be equivalent to each other. We will here mostly be working with the model given by quasicategories. It is also possible to work with ∞-categories in an entirely model-independent way: for example, see [RV19].

2.2. **Definition.** A simplicial set K is a *quasicategory* if, for every i such that $0 < i < n$, any map $f : \Lambda^n_i \to K$ can be extended to a map $\tilde{f} : \Delta^n \to K$.

2.3. *Remark.* The definition says we can only fill in the "inner horns" in K. If we also asked to be able to fill in the "outer horns" Λ^n_0 and Λ^n_n, then we would get a *Kan complex*, which is a model for an ∞-*groupoid*: being able to fill in the outer horns gives an inverse (up to contractible

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choice) for any 1-morphism. On the other hand, if we asked to have *unique* fillers for the inner horns, then we would get (the nerve of) a 1-category.

Recall that the intuition for ∞-categories is that we should be able to think of them as categories enriched in topological spaces. We should therefore be able to recover a 1-category from an ∞-category $\mathscr{C}$ by taking the path components of each mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y)$.

2.4. **Definition.** Let $\mathscr{C}$ be a quasicategory. Its *homotopy category* $h\mathscr{C}$ is the 1-category whose objects are the 0-simplices of $\mathscr{C}$ and whose morphisms are homotopy classes of 1-simplices of $\mathscr{C}$: we declare two 1-simplices $f, g : x \to y$ to be homotopic if there is a 2-simplex in $\mathscr{C}$ of the following form:

$$\begin{array}{ccccc} & & x & & \\ & /\!\!/ & & \searrow^{f} & \\ x & & \xrightarrow[g]{} & & y \end{array}$$

This is an equivalence relation on the 1-simplices of $\mathscr{C}$.

2.5. *Remark.* In the previous definition, we chose to have two vertices labelled "x" rather than two vertices labelled "y". In a general simplicial set, these two choices generate different equivalence relations; in a quasicategory, they are both already equivalence relations, and are moreover the same equivalence relation.

We have already mentioned the mapping spaces $\mathrm{Map}_{\mathscr{C}}(x, y)$ of an ∞-category, and here give an explicit model for their homotopy type.

2.6. **Definition.** Let $\mathscr{C}$ be a quasicategory. The mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ is the homotopy type of the strict pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(x, y) & \longrightarrow & \mathscr{C}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow \\ \{(x, y)\} & \longrightarrow & \mathscr{C} \times \mathscr{C} \end{array}$$

taken in the 1-category of simplicial sets.

2.7. *Remark.* We can make a very similar definition for $\mathrm{Map}_{\mathscr{C}}(x, y, z)$, namely

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(x, y, z) & \longrightarrow & \mathscr{C}^{\Delta^2} \\ \downarrow & \lrcorner & \downarrow \\ \{(x, y, z)\} & \longrightarrow & \mathscr{C} \times \mathscr{C} \times \mathscr{C} \end{array}$$

The simplicial set $\mathrm{Map}_{\mathscr{C}}(x, y, z)$ comes equipped with maps

$$\mathrm{Map}_{\mathscr{C}}(y, z) \times \mathrm{Map}_{\mathscr{C}}(x, y) \overset{\sim}{\twoheadleftarrow} \mathrm{Map}_{\mathscr{C}}(x, y, z) \to \mathrm{Map}_{\mathscr{C}}(x, z).$$

The left-facing map is a "trivial Kan fibration", which in particular implies that it has a unique (up to contractible choice) section. Making such a choice, we obtain a composition map

$$\mathrm{Map}_{\mathscr{C}}(y, z) \times \mathrm{Map}_{\mathscr{C}}(x, y) \to \mathrm{Map}_{\mathscr{C}}(x, z),$$

i.e. there is a unique (up to contractible choice) way of composing morphisms in a quasicategory.

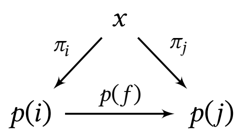
We can now define the simplest (co)limits, namely initial and terminal objects.

2.8. **Definition.** Let $\mathscr{C}$ be a quasicategory. An object $x \in \mathscr{C}$ is *initial* if, for every $y \in \mathscr{C}$, the mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ is contractible. Dually, an object $x \in \mathscr{C}$ is *terminal* if, for every $y \in \mathscr{C}$, the mapping space $\mathrm{Map}_{\mathscr{C}}(y, x)$ is contractible.

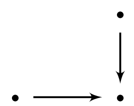
2.9. *Remark.* In the 1-categorical setting, an initial object is unique up to unique isomorphism, if it exists. The appropriate ∞-categorical analogue is that the sub-simplicial set of $\mathscr{C}$ spanned by the initial objects is either empty or forms a contractible Kan complex. When we said that composition was "unique up to contractible choice" above, this was the type of uniqueness that was meant.

We can define a limit to be a terminal object of a certain ∞-category, which we will now define. This mirrors the 1-categorical definition of a limit as a terminal cone:

2.10. *Reminder.* We briefly recap the definition of a limit in a 1-category. Let $p : J \to C$ be a diagram of shape J in a 1-category C. A *cone* over p is an object $x \in C$ together with morphisms $\pi_j : x \to p(j)$ for every $j \in J$, such that the diagram



commutes in C for every morphism $f : i \to j$ in J. A morphism of cones is a morphism $x \to x'$ in C that commutes with the projection maps π_j and π'_j. In this way, we get a category $C_{/p}$ of cones over p, and a limit of p is a terminal object of this category. If this example doesn't make sense to you, then work through the example of a pullback, where we take J to be the diagram

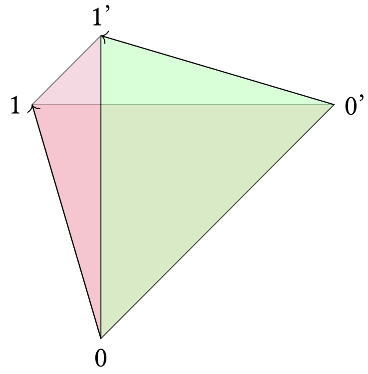


2.11. **Definition.** Given two simplicial sets S and T, we define their *join* to be the simplicial set whose value on a finite, non-empty, linearly ordered set J is

$$(S \star T)(J) := \coprod_{I \cup I' = J} S(I) \times T(I').$$

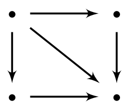
Here the coproduct ranges over decompositions of J into disjoint sets $I \cup I'$ such that every element of I is less than every element of I'. We have also adopted the convention that $S(\varnothing)$ is a point, so that the 0-simplices of $S \star T$ are the disjoint union of the 0-simplices of S and the 0-simplices of T.

2.12. *Example.* The join of two copies of Δ^1 is Δ^3:



Of the six 1-simplices in Δ^3, two of them are 1-simplices in the two copies of Δ^1, and the remaining four 1-simplices join a 0-simplex of the first copy of Δ^1 to a 0-simplex of the second copy of Δ^1.

2.13. *Example.* The join of Δ^0 with a simplicial set K is the same as attaching an "initial" vertex to K. We will denote this simplicial set by $K^{\triangleleft}$. Dually, $K \star \Delta^0$ is attaching a terminal vertex to K and will be denoted $K^{\triangleright}$. With this notation, we have



$$\cong \Delta^1 \times \Delta^1 \cong (\Lambda^2_0)^{\triangleright} \cong (\Lambda^2_2)^{\triangleleft}.$$

This fact will be relevant later, when we consider pullbacks and pushouts.

2.14. **Definition.** Let $p : K \to \mathscr{C}$ be a diagram of shape K in a quasicategory $\mathscr{C}$. (Here K can be an arbitrary simplicial set.) Write $\hom(X, Y)$ for the set of morphisms in the 1-category of simplicial sets. The *slice quasicategory over p* is the quasicategory $\mathscr{C}_{/p}$ with n-simplices

$$(\mathscr{C}_{/p})_n := \hom_p(\Delta^n \star K, \mathscr{C}),$$

where the right-hand side denotes the subset of $\hom(\Delta^n \star K, \mathscr{C})$ consisting of those morphisms whose restriction to K is p. Dually, the *slice quasicategory under p* is the quasicategory $\mathscr{C}_{p/}$ with n-simplices

$$(\mathscr{C}_{p/})_n := \hom_p(K \star \Delta^n, \mathscr{C}),$$

where the right-hand side now denotes the subset of $\hom(K \star \Delta^n, \mathscr{C})$ consisting of those morphisms whose restriction to K is p.

2.15. *Remark.* The fact that $\mathscr{C}_{/p}$ and $\mathscr{C}_{p/}$ are quasicategories (rather than just simplicial sets) is not completely trivial; see [HTT, Corollary 2.1.2.2] for a proof.

2.16. *Remark.* The objects of $\mathscr{C}_{/p}$ are the ∞-categorical analogues of cones over the diagram p.

2.17. **Definition.** Let $p : K \to \mathscr{C}$ be a diagram of shape K in a quasicategory $\mathscr{C}$. A *limit* of p is a terminal object of $\mathscr{C}_{/p}$, and a *colimit* of p is an initial object of $\mathscr{C}_{p/}$.

2.18. *Example.* We consider the case of pullbacks and pushouts. We earlier remarked that $\Delta^1 \times \Delta^1 \cong (\Lambda^2_0)^{\triangleright} \cong (\Lambda^2_2)^{\triangleleft}$. These isomorphisms mean that if we specify a square in a quasicategory $\mathscr{C}$ (that is, if we give a map of simplicial sets $\Delta^1 \times \Delta^1 \to \mathscr{C}$) then it makes sense to both ask whether it is a pushout, considering $\Delta^1 \times \Delta^1$ as $(\Lambda^2_0)^{\triangleright}$, and to ask whether it is a pullback, considering $\Delta^1 \times \Delta^1$ as $(\Lambda^2_2)^{\triangleleft}$.

Suppose our square $s : \Delta^1 \times \Delta^1 \to \mathscr{C}$ is indicated by

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

and we wish to determine whether it's a pullback. We first need to consider s as a zero-simplex of $\mathscr{C}_{/p}$, where $p : \Lambda^2_2 \to \mathscr{C}$ is the diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

Then for every square $t : \Delta^1 \times \Delta^1 \to \mathscr{C}$ of the form

$$\begin{array}{ccc} A' & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D, \end{array}$$

we need to check that the mapping space $\mathrm{Map}_{\mathscr{C}_{/p}}(t, s)$ is contractible. As with many things in ∞-category theory, this is rarely something that you will check directly. Some techniques for computing (co)limits include using an abstract argument to compare a (co)limit with an already-constructed object and showing that they are equivalent, or finding a model category that models your ∞-category and computing the homotopy (co)limit in the model category.

3. Stable ∞-categories

In this section, we'll move on to talking about stable ∞-categories. Our two guiding examples will be the (unbounded) derived category of an abelian category and the category of spectra. We'll start by asking what these two examples have in common, then we'll formalise this into a definition that will end up being the definition of a stable ∞-category. This section mostly follows [HA, §1.1.1].

3.1. *Remark.* For the purposes of motivating the definition of a stable ∞-category, it shouldn't matter too much whether you read the below discussion thinking of the examples as ∞-categories, as triangulated categories, or just have a vague idea of what the objects and morphisms in the categories "should" be. However, for the sake of being concrete about what we're discussing, let's define the two examples as ∞-categories:

Let $\mathcal{A}$ be a Grothendieck abelian category, *i.e.* a locally presentable abelian category in which the small filtered colimit of a collection of short exact sequences is still a short exact sequence. Let $\mathrm{Ch}(\mathcal{A})$ denote the 1-category of chain complexes in $\mathcal{A}$. We can put a model structure on $\mathrm{Ch}(\mathcal{A})$, whose cofibrations are the levelwise monomorphisms and whose weak equivalences are the quasi-isomorphisms. We define the (unbounded) derived ∞-category of $\mathcal{A}$ to be

$$\mathcal{D}(\mathcal{A}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})^{\circ}),$$

the differential graded nerve (see [HA, §1.3.1]) of the bifibrant objects with respect to this model structure.

In order to define spectra, we first want to define pointed spaces. Informally, we want the ∞-category of spaces to be the "spaces with weak equivalences inverted". More precisely, we can define $\mathcal{S}$, the quasicategory of spaces, to be the simplicial nerve of the simplicially enriched category of Kan complexes. The quasicategory of pointed spaces $\mathcal{S}_*$ is then defined to be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{S})$ consisting of those functors that send $0 \in \Delta^1$ to a terminal object of $\mathcal{S}$. We can now define the ∞-category of spectra as the limit

$$\mathrm{Sp} \coloneqq \lim(\dots \to \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$

taken in Cat_∞, the ∞-category of (small) ∞-categories (see [HTT, §3.0.0.1] for a definition of Cat_∞). Here Ω denotes the loop space functor, which we will formally define in Proposition 3.13.

The first thing that Sp and $\mathcal{D}(\mathcal{A})$ have in common is a zero object:

3.2. **Definition.** A *zero object* in an ∞-category is an object that is both initial and terminal. An ∞-category with a zero object is said to be *pointed*.

3.3. *Example.* In Sp, a zero object is the spectrum that has a point in every level. More generally, the zero objects are those whose homotopy groups all vanish.

In $\mathcal{D}(\mathcal{A})$, the zero objects are the acyclic complexes. That is, a complex is a zero object if all of its homology groups vanish.

3.4. *Remark.* Notice that here we can see that initial (and terminal) objects in an ∞-category need not be unique, but only unique up to a contractible space of choices. This non-uniqueness is something that already happens in 1-categories: for example, any 1-point set is terminal in the category of sets, but $\{0\}$ and $\{1\}$ are two different (yet isomorphic) terminal objects. However, I personally find it significantly more surprising that the acyclic complex

$$\dots \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to \dots$$

is a zero object in $\mathcal{D}(\mathcal{A})$!

3.5. *Remark.* In a pointed ∞-category, we get a zero morphism between any two objects x and y by composing "the" morphisms $x \to 0$ and $0 \to y$. The zero morphism is unique up to a contractible space of choices.

The second thing that Sp and $\mathcal{D}(\mathcal{A})$ have in common is a concept of (co)fibres or (co)kernels:

3.6. *Example.* If we have a chain map $f : C \to D$ between chain complexes, then we get an associated short exact sequence

$$0 \to C \to \mathrm{cyl}(f) \to \mathrm{cone}(f) \to 0,$$

where $\mathrm{cyl}(f)$ is a chain complex quasi-isomorphic to D. We can often say interesting things about the map f by studying $\mathrm{cone}(f)$; for example, there's a long exact sequence relating $\mathrm{H}_*(C)$, $\mathrm{H}_*(D)$ and $\mathrm{H}_*(\mathrm{cone}(f))$, so $\mathrm{cone}(f)$ is acyclic if and only if f is a quasi-isomorphism. We can view $\mathrm{cone}(f)$ as a "cokernel" or "cofibre" of f.

Similarly, if we have a map $f : X \to Y$ between spectra, then we get a fibre sequence

$$\mathrm{fib}(f) \to X \xrightarrow{f} Y,$$

which comes with an associated long exact sequence on homotopy groups. There is also a concept of a cofibre sequence, and it turns out that fibre sequences are exactly the same as cofibre sequences; that is,

$$\mathrm{cofib}(\mathrm{fib}(f) \to X) \simeq Y.$$

3.7. **Definition.** Let $\mathscr{C}$ be a pointed quasicategory. A *triangle* in $\mathscr{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathscr{C}$ of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

where 0 denotes the zero object of $\mathscr{C}$.

3.8. *Remark.* More explicitly, a triangle is the following data:

(i) Two 1-simplices $X \xrightarrow{f} Y \xrightarrow{g} Z$,
(ii) a 2-simplex

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

witnessing a composite h of f and g, and
(iii) a 2-simplex

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{h} & \\ 0 & \longrightarrow & Z \end{array}$$

witnessing a null-homotopy of h.

We don't draw the diagonal 1-simplex or the 2-simplices when we represent a triangle, but it is important to remember that they're there!

3.9. *Remark.* A diagram $\Delta^1 \times \Delta^1 \to \mathscr{C}$ of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

determines a unique triangle (though isn't one itself): we just need to precompose with the map $\Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^1$ that swaps the two factors.

3.10. **Definition.** Let $\mathscr{C}$ be a pointed quasicategory, and let $p : \Delta^1 \times \Delta^1 \to \mathscr{C}$ be the triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z. \end{array}$$

We say that p is a *fibre sequence* if p is a pullback diagram (in which case we also say that p is a *fibre of* g). Dually, we say that p is a *cofibre sequence* if p is a pushout diagram (in which case we also say that p is a *cofibre of* f).

3.11. **Definition.** An ∞-category $\mathscr{C}$ is *stable* if

(i) there exists a zero object of $\mathscr{C}$,
(ii) every morphism of $\mathscr{C}$ has a fibre and a cofibre, and
(iii) a triangle is a fibre sequence if and only if it is a cofibre sequence.

3.12. *Remark.* Stability is a property of an ∞-category, not structure. If you're not familiar with the distinction between properties and structure, consider the example of sets, groups, and abelian groups.

If I give you a set X, then it doesn't make sense to ask "Is X a group?" without also specifying the function $m : X \times X \to X$ that would give the group multiplication. In other words, being a group isn't an intrinsic property of the underlying set of the group.

On the other hand, if I give you a group G, it *does* make sense to ask "Is G abelian?". I don't need to specify any more data in order to answer this question; being abelian (or not) is an intrinsic property of a group.

Some cases are more subtle: if I give you an abelian group A, is "being the additive group of an $\mathbb{F}_p$-vector space" property or structure? At first glance, it looks like it's structure, because I have to tell you how scalar multiplication by an element of $\mathbb{F}_p$ acts on A. However, if A is an $\mathbb{F}_p$-vector space, the scalar multiplication is in fact going to be uniquely determined by the abelian group structure on A, since

$$n \cdot v = (1 + \ldots + 1) \cdot v = v + \ldots + v.$$

So being the additive group of an $\mathbb{F}_p$-vector space is actually a property of an abelian group!

Finally, why do we care? There's an analogy to be made with linear algebra: it's often easier to work coordinate-free than to choose bases. Similarly, if you know that something's a property, it's often easier to formulate cleaner abstract arguments. However, the takeaway from this isn't necessarily that "properties are good and structure is bad": remember that one of the insights of ∞-category theory is that the property "X is isomorphic to Y" is almost always less useful than the structure "an isomorphism $\theta : X \to Y$".

3.13. **Proposition.** *Let $\mathscr{C}$ be a pointed ∞-category. The following are equivalent:*

(i) $\mathscr{C}$ is stable.

(ii) $\mathscr{C}$ has finite colimits and the "suspension" functor $\Sigma : \mathscr{C} \to \mathscr{C}$ determined by the existence of pushout squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0' & \longrightarrow & \Sigma X \end{array}$$

is an equivalence. (Here 0 and $0'$ denote zero objects of $\mathscr{C}$.)

(iii) $\mathscr{C}$ has finite limits and the "loop space" functor $\Omega : \mathscr{C} \to \mathscr{C}$ determined by the existence of pullback squares

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0' & \longrightarrow & X \end{array}$$

is an equivalence.

Proof. See [HA, Corollary 1.4.2.27]. □

4. TRIANGULATED CATEGORIES

As mentioned in the introduction, one possible approach to capturing the "stable" properties of the derived category or the category of spectra is to use triangulated categories. In this section, we define triangulated categories and show that the homotopy category of a stable

∞-category comes with a triangulated structure (in fact, at least two such structures). This section follows [HA, §1.1.2].

4.1. *Remark.* Let $\mathcal{D}$ be a 1-category that has finite biproducts. We can give each hom-set $\hom_{\mathcal{D}}(X, Y)$ the structure of an abelian monoid by defining

$$f + g := X \xrightarrow{\Delta_X} X \times X \simeq X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \simeq Y \amalg Y \xrightarrow{\nabla_Y} Y.$$

4.2. **Definition.** A 1-category $\mathcal{D}$ is *additive* if it has finite biproducts and the abelian monoid structure on $\hom_{\mathcal{D}}(X, Y)$ that was defined in Remark 4.1 is in fact an abelian group.

4.3. *Remark.* Note that being a group is a property of a monoid and having finite biproducts is a property of a category, so being additive is also a property.

4.4. **Definition.** A *triangulated category* is an additive 1-category $\mathcal{D}$ with an equivalence $\Sigma : \mathcal{D} \to \mathcal{D}$ and a class of *distinguished triangles*, each of which is a diagram of the form

$$X \to Y \to Z \to \Sigma X.$$

These data should satisfy:

TR1: (i) For every object X, the triangle

$$X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma X$$

is distinguished.

(ii) For every morphism $u : X \to Y$, there exists an object Z and a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to \Sigma X.$$

(iii) The class of distinguished triangles is closed under isomorphism.

TR2: The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if the "rotated" triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

TR3: Given a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X', \end{array}$$

such that the rows are distinguished triangles and $vf = f'u$, there is a (not necessarily unique!) dashed arrow making the diagram commute.

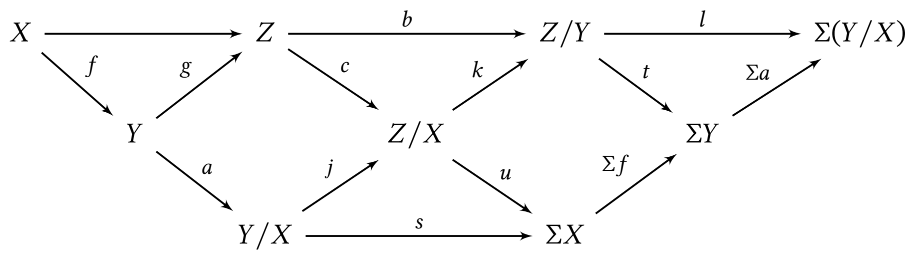
TR4: Suppose we have morphisms $f : X \to Y$ and $g : Y \to Z$. Let

$$X \xrightarrow{f} Y \xrightarrow{a} Y/X \xrightarrow{s} \Sigma X,$$
$$Y \xrightarrow{g} Z \xrightarrow{b} Z/Y \xrightarrow{t} \Sigma Y, \quad \text{and}$$
$$X \xrightarrow{gf} Z \xrightarrow{c} Z/X \xrightarrow{u} \Sigma X$$

be distinguished triangles, where the symbols Y/X, Z/Y and Z/X are merely evocative names for objects of $\mathcal{D}$. (We know that such triangles exist by (TR1).) For any such triangles, there exists a distinguished triangle

$$Y/X \xrightarrow{j} Z/X \xrightarrow{k} Z/Y \xrightarrow{l} \Sigma(Y/X)$$

such that the following diagram commutes:



4.5. *Remark.* Part of the point of writing all that out is to allow you to appreciate how compact the definition of a stable ∞-category is! We won't be using triangulated categories much, so don't worry if the axioms seem a bit obscure. The last axiom, (TR4), is known as the "octahedral axiom". This is because the large diagram appearing there can also be represented as an octahedron.

4.6. **Proposition.** *If $\mathscr{C}$ is a stable ∞-category, then $h\mathscr{C}$ can be given a triangulated structure.*

We won't prove this completely, but in the rest of the section will at least indicate how to define a triangulated structure and why (TR2) and (TR4) hold. This is pretty long-winded, but almost everything we're doing is abstract nonsense playing around with universal properties of pushouts and pullbacks.

In Proposition 3.13, we indicated how to construct the functors Ω and Σ as pullbacks and pushouts respectively. Our first check is that these two functors are inverse equivalences; this follows because fibre sequences are the same as cofibre sequences in $\mathscr{C}$. Indeed, if $X \in \mathscr{C}$ then we have a cofibre sequence

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0' & \longrightarrow & \Sigma X. \end{array}$$

This is also a fibre sequence, and so (by definition of Ω) we have $\Omega\Sigma X \simeq X$. A dual argument shows that $\Sigma\Omega X \simeq X$.

We next wish to show that $h\mathscr{C}$ is additive. In particular, we need coproducts in $h\mathscr{C}$, and we will achieve this by constructing coproducts in $\mathscr{C}$. (We hereby provide part of the proof of Proposition 3.13.) We claim that

$$X \amalg Y \simeq \operatorname{cofib}(\Omega X \xrightarrow{0} Y).$$

Indeed, we have a diagram

$$\begin{array}{ccccc} \Omega X & \longrightarrow & 0 & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ 0' & \longrightarrow & X & \longrightarrow & C \end{array}$$

where we have constructed the left-hand pushout by taking the cofibre of $\Omega X \to 0$, and the outer pushout by taking the cofibre of the zero morphism $\Omega X \to Y$. We have written C for this latter cofibre, and the map $X \to C$ is induced by the universal property. By [HTT, Lemma

4.4.2.1], the right-hand square is also a pushout. We then observe that the right-hand square is exactly the pushout that gives the universal property of $X \amalg Y$, proving the claim.

This coproduct is also a product: indeed, by a dual argument,

$$X \times Y \simeq \operatorname{fib}(X \xrightarrow{0} \Sigma Y).$$

Let us write F for this fibre. Both Ω and fib are defined by pullbacks, so they commute. We can apply Ω to get a fibre sequence

$$\begin{array}{ccc} \Omega F & \longrightarrow & \Omega X \\ \downarrow & \lrcorner & \downarrow {\scriptstyle 0} \\ 0 & \longrightarrow & Y \end{array}$$

which (since $\mathscr{C}$ is stable) is also a cofibre sequence. Combining this with the cofibre sequence defining C gives us

$$\begin{array}{ccccc} \Omega F & \longrightarrow & \Omega X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 0} & \ulcorner & \downarrow \\ 0' & \longrightarrow & Y & \longrightarrow & C \end{array}$$

and examining the exterior pushout shows that

$$C \simeq \Sigma \Omega F \simeq F.$$

To finish our proof that $h\mathscr{C}$ is additive, we need to check that the abelian monoid structure on $\hom_{h\mathscr{C}}(X, Y)$ defined in Remark 4.1 is in fact an abelian group structure. For any $X, Y \in \mathscr{C}$, we have $Y \simeq \Omega^n \Sigma^n Y$, so

$$\begin{aligned} \operatorname{Map}_{\mathscr{C}}(X, Y) &\simeq \operatorname{Map}_{\mathscr{C}}(X, \Omega^n \Sigma^n Y) \\ &\simeq \Omega^n \operatorname{Map}_{\mathscr{C}}(X, \Sigma^n Y) \end{aligned}$$

Note that in ∞-categories (homotopy) limits commute with $\operatorname{Map}_{\mathscr{C}}(X, -)$ in the same way that limits commute with $\hom_{\mathcal{D}}(X, -)$ in 1-categories, which is why we can move the Ω^n to the front of the expression. This implies that $\operatorname{Map}_{\mathscr{C}}(X, Y)$ is an infinite loop space, so $\pi_0 \operatorname{Map}_{\mathscr{C}}(X, Y)$ is an abelian group under loop sum.

We now have two potentially different abelian monoid structures on $\pi_0 \operatorname{Map}_{\mathscr{C}}(X, Y)$: one coming from loop sum and one coming from binary products. We know that the first of these is an abelian group, so we will show that the two products are the same. In order to do this, we can use the Eckmann–Hilton argument (see [Wik] or [Spa66, Chapter 1, §6, Theorem 8], where the hypothesis that the two operations share the same identity is unnecessary). The hypothesis that the "binary product" sum is compatible with "loop" sum follows from observing that the "binary product" sum on $\pi_0 \Omega \operatorname{Map}_{\mathscr{C}}(X, \Sigma Y)$ is induced by

$$\operatorname{Map}_{\mathscr{C}}(X, \Sigma Y) \times \operatorname{Map}_{\mathscr{C}}(X, \Sigma Y) \xrightarrow{\oplus} \operatorname{Map}_{\mathscr{C}}(X \oplus X, \Sigma Y \oplus \Sigma Y) \xrightarrow{\Delta^* \circ \nabla_*} \operatorname{Map}_{\mathscr{C}}(X, \Sigma Y)$$

and that this composition preserves the basepoint.

4.7. *Remark.* As indicated above, we can define a *mapping spectrum* $\operatorname{map}_{\mathscr{C}}(X, Y)$ in $\mathscr{C}$. As an Ω-spectrum, the n^{th} space is given by

$$X_n \coloneqq \operatorname{Map}_{\mathscr{C}}(X, \Sigma^n Y).$$

The zeroth space X_0 is $\operatorname{Map}_{\mathscr{C}}(X, Y)$, so we have

$$\Omega^\infty \operatorname{map}_{\mathscr{C}}(X, Y) \simeq \operatorname{Map}_{\mathscr{C}}(X, Y).$$

In fact, every stable ∞-category is canonically enriched in Sp (see [GH15, Example 7.4.14]).

Finally, we define what the distinguished triangles in $h\mathscr{C}$ are.

4.8. **Definition.** A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $h\mathscr{C}$ is distinguished if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$ of the form

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \tilde{g}} & \ulcorner & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow[\tilde{h}]{} & W \end{array}$$

such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{h}} & W \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \simeq} \\ & & \Sigma X \end{array}$$

Note that the first diagram implies that W is another model for ΣX, so there is a canonical (up to contractible choice) equivalence $W \simeq \Sigma X$; this equivalence is the vertical map in the second diagram.

4.9. *Remark.* Note that we've made a choice to have triangles of the form $\Delta^1 \times \Delta^2 \to \mathscr{C}$ rather than triangles of the form $\Delta^2 \times \Delta^1 \to \mathscr{C}$. Either choice works, but you have to be consistent: the two choices give rise to two different triangulated structure on $h\mathscr{C}$.

We have now defined all the structure needed to check whether $h\mathscr{C}$ is a triangulated category. We finish the section by saying a few words about why two of the more tricky axioms for triangulated categories hold. We start with a proposition about the abelian group structure on $\hom_{h\mathscr{C}}(X, Y)$. Note that a square

$$\begin{array}{ccc} X & \xrightarrow{f} & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow \\ 0' & \longrightarrow & W \end{array}$$

represents an element of $\hom_{h\mathscr{C}}(\Sigma X, W)$, and in fact all elements of $\hom_{h\mathscr{C}}(\Sigma X, W)$ lift to such a square.

4.10. **Proposition.** *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow \\ 0' & \longrightarrow & W \end{array}$$

represents the morphism $\theta \in \hom_{h\mathscr{C}}(\Sigma X, W)$, then

$$\begin{array}{ccc} X & \xrightarrow{f'} & 0' \\ \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & W \end{array}$$

represents $-\theta$ with respect to the abelian group structure.

Proof. See [HA, Lemma 1.1.2.10] for a proof; here we give an informal argument to explain why the claim is plausible.

Given two spaces X and Y, the group structure on $\pi_0 \operatorname{Map}(\Sigma X, Y)$ is given by concatenating along the "cone" coordinate in ΣX. This implies that the inverse of a map θ should be determined by reversing the direction of the cone coordinate in θ; that is, we swap the two cone points in ΣX.

Swapping the cone points is exactly what transposing our diagram achieves: we can view

$$\begin{array}{ccc} X & \xrightarrow{f} & 0 \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow \\ 0' & \longrightarrow & W \end{array}$$

as a map $X \to W$, given by the omitted diagonal 1-simplex, along with two different null-homotopies of this map. The null-homotopies define the extension to a map $\Sigma X \to W$, and if we transpose the diagram then we swap the cone points in ΣX. □

4.11. **Proposition.** *A triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} \Sigma X$$

in $h\mathscr{C}$ is distinguished if and only if we can lift it to a diagram $p : \Delta^2 \times \Delta^1 \to \mathscr{C}$ of the form

$$\begin{array}{ccc} X & \longrightarrow & 0' \\ {\scriptstyle \tilde f}\big\downarrow & \ulcorner & \big\downarrow \\ Y & \xrightarrow[\tilde g]{} & Z \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \tilde h} \\ 0 & \longrightarrow & W \end{array}$$

such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde h} & W \\ & {\scriptstyle h}\searrow & \big\downarrow{\scriptstyle \simeq} \\ & & \Sigma X \end{array}$$

The vertical map in the second diagram is the canonical equivalence $W \simeq \Sigma X$ induced by the first diagram.

Proof. In order to show that we have a triangle, we need to exhibit a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$. We take the transpose of the diagram p. We now come to a subtle point: in the transposed diagram, while W is still equivalent to ΣX, we have changed the sign of the canonical map $W \simeq \Sigma X$. This is precisely because the universal property of a colimit also takes the structure maps into account, and when we transpose the diagram, we end up in the situation of Proposition 4.10. This is where the choice that we commented on in Remark 4.9 matters.

Putting all this together, we have a diagram $\overline{p} : \Delta^1 \times \Delta^2 \to \mathscr{C}$

$$\begin{array}{ccccc} X & \xrightarrow{\tilde f} & Y & \longrightarrow & 0 \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \tilde g} & \ulcorner & \big\downarrow \\ 0' & \longrightarrow & Z & \xrightarrow[\tilde h]{} & W \end{array}$$

such that

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{h}} & W \\ & \searrow^{-h} & \downarrow{\simeq} \\ & & \Sigma X \end{array}$$

where the vertical map in the last diagram is the canonical equivalence induced by the diagram $\overline{p}$, *i.e* the negative of the canonical equivalence induced by the diagram p. □

Using this, we can show (TR2), *i.e.* how to rotate distinguished triangles in $h\mathscr{C}$. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a distinguished triangle in $h\mathscr{C}$, so we have a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow{\tilde{g}} & \ulcorner & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow[\tilde{h}]{} & W \end{array}$$

We can extend this diagram further by taking the cofibre of $\tilde{h}$:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow{\tilde{g}} & \ulcorner & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow[\tilde{h}]{} & W \\ & & \downarrow & \ulcorner & \downarrow{\tilde{k}} \\ & & 0'' & \longrightarrow & U \end{array} \tag{4.12}$$

By the same argument as previously, we have canonical equivalences $W \simeq \Sigma X$ and $U \simeq \Sigma Y$. These fit into a commutative square

$$\begin{array}{ccc} W & \xrightarrow{\simeq} & \Sigma X \\ \downarrow{\tilde{k}} & & \downarrow{\Sigma \tilde{f}} \\ U & \xrightarrow{\simeq} & \Sigma Y \end{array}$$

by definition of the functor Σ. By Proposition 4.11, the two right-hand squares of Diagram 4.12 witness that the rotated triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished. In order to show the converse implication, you first apply Ω^2 to the triangle then rotate it (as above) five times.

Finally, we discuss (TR4), the octahedral axiom. Recall that we have morphisms $f : X \to Y$ and $g : Y \to Z$ that we complete to distinguished triangles

$$\begin{aligned} &X \to Y \to Y/X \to \Sigma X, \\ &Y \to Z \to Z/Y \to \Sigma Y, \quad \text{and} \\ &X \to Z \to Z/X \to \Sigma X \end{aligned}$$

and we wish to construct a distinguished triangle

$$Y/X \to Z/X \to Z/Y \to \Sigma(Y/X).$$

By repeatedly taking cofibres like previously, we can construct a large diagram

$$\begin{array}{ccccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0' & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & \Sigma X & \longrightarrow & 0'' \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0''' & \longrightarrow & Z/Y & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma(Y/X)
\end{array}$$

where every square is a pushout. I've been slightly less careful than before, in that here I have written ΣX instead of giving a new name to the object appearing in that position in the diagram. The terms appearing in red are the vertices of a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$ that witnesses that the triangle in which we are interested is distinguished. All of the equations that we need to hold can be read off from this diagram.

4.13. *Remark.* We have shown that the homotopy category of any stable ∞-category has a triangulated structure. In fact, all of the triangulated categories that occur in nature (spectra, the derived category, the stable module category, ...) are known to be the homotopy category of a stable ∞-category. There are examples of triangulated categories that are explicitly designed not to be the homotopy category of a stable ∞-category (see [MSS07]), but we should probably view this as a failure of the definition of a triangulated category rather than the definition of a stable ∞-category.

5. t-structures and the filtered object spectral sequence

Given a filtered chain complex

$$\ldots \subseteq F_{p-1} \subseteq F_p \subseteq \ldots \subseteq C$$

we get a spectral sequence

$$E^1_{p,q} = \mathrm{H}_{p+q}(F_p/F_{p-1}) \Rightarrow \mathrm{H}_{p+q}(C)$$

whose convergence is conditional on finiteness properties of the filtration. This spectral sequence arises from the short exact sequences of chain complexes

$$0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0.$$

We're going to generalise this spectral sequence to any stable ∞-category with a certain structure, called a "t-structure". This section mostly follows [HA, §1.2.1].

5.1. **Definition.** Let $\mathscr{C}$ be a stable ∞-category. A *t-structure* on $\mathscr{C}$ is a pair $(\mathscr{C}_{\geq 0}, \mathscr{C}_{<0})$ of full subcategories (closed under equivalences in $\mathscr{C}$) such that:

(i) for any $X \in \mathscr{C}_{\geq 0}$ and $Y \in \mathscr{C}_{<0}$, we have $\mathrm{Map}_{\mathscr{C}}(X, Y) \simeq *$,
(ii) $\Sigma\mathscr{C}_{\geq 0} \subseteq \mathscr{C}_{\geq 0}$ and $\Omega\mathscr{C}_{<0} \subseteq \mathscr{C}_{<0}$, and
(iii) every $X \in \mathscr{C}$ fits into a fibre sequence

$$\begin{array}{ccc}
X_{\geq 0} & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow \\
0 & \longrightarrow & X_{<0}
\end{array}$$

where $X_{\geq 0} \in \mathscr{C}_{\geq 0}$ and $X_{<0} \in \mathscr{C}_{<0}$.

5.2. *Remark.* This definition is different from the one given in class (which was [HA, Definition 1.2.1.4]); that definition first defined a t-structure on a triangulated category, then defined a t-structure on a stable ∞-category as a t-structure on its homotopy category. The above definition is entirely inside the ∞-category itself, and is taken from [FL16, Definition 15].

5.3. *Remark.* $\mathscr{C}_{\geq 0}$ and $\mathscr{C}_{<0}$ determine each other. For example, $Y \in \mathscr{C}_{<0}$ if and only if $\mathrm{Map}_{\mathscr{C}}(X, Y)$ is contractible for all $X \in \mathscr{C}_{\geq 0}$.

5.4. **Proposition.** *We have two pairs of adjoint functors*

$$\mathscr{C}_{\geq 0} \underset{\tau_{\geq 0}}{\overset{\text{incl}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{C} \underset{\text{incl}}{\overset{\tau_{<0}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{C}_{<0}$$

where, using the notation of Definition 5.1(iii), we have $\tau_{\geq 0} X = X_{\geq 0}$ *and* $\tau_{<0} X = X_{<0}$.

5.5. *Remark.* I use the word "adjunction" in the above proposition to mean an adjunction of ∞-categories. As we only care about the existence of the functors $\tau_{\leq 0}$ and $\tau_{\geq 0}$, it's not very important to understand the definition of such an adjunction if you're willing to take on faith that they "work as you expect from 1-category theory": right adjoints preserve limits, left adjoints preserve colimits, there is an equivalence of mapping spaces

$$\mathrm{Map}_{\mathscr{D}}(LX, Y) \simeq \mathrm{Map}_{\mathscr{C}}(X, RY)$$

and so on.

If you're interested, however, one precise definition is as follows: an adjunction between $\mathscr{C}$ and $\mathscr{D}$ is a morphism $p : K \to \Delta^1$ of simplicial sets that is simultaneously a cartesian and a cocartesian fibration, along with equivalences $p^{-1}(0) \simeq \mathscr{C}$ and $p^{-1}(1) \simeq \mathscr{D}$. A cocartesian lift of the morphism $0 \to 1$ in Δ^1 gives you the left adjoint $\mathscr{C} \to \mathscr{D}$ and a cartesian lift of the same morphism gives you the right adjoint $\mathscr{D} \to \mathscr{C}$.

In ∞-category theory, we almost always construct adjunctions by use of the adjoint functor theorem.

5.6. *Remark.* We will often identify $\mathscr{C}_{\geq 0}$ with its image in $\mathscr{C}$, and think of $\tau_{\geq 0}$ as a functor $\mathscr{C} \to \mathscr{C}_{\geq 0} \to \mathscr{C}$.

5.7. **Definition.** We write $\mathscr{C}_{\geq n} \coloneqq \Sigma^n \mathscr{C}_{\geq 0}$ and $\mathscr{C}_{<n} \coloneqq \Sigma^n \mathscr{C}_{<0}$. We similarly get truncation functors $\tau_{\geq n} : \mathscr{C} \to \mathscr{C}_{\geq n}$ and $\tau_{<n} : \mathscr{C} \to \mathscr{C}_{<0}$ that are adjoint to the inclusions of these subcategories. We also write $\mathscr{C}_{>n} \coloneqq \mathscr{C}_{\geq n+1}$ and $\mathscr{C}_{\leq n} \coloneqq \mathscr{C}_{<n+1}$.

5.8. *Example.* For $\mathscr{C} = \mathcal{D}(R)$, we have a t-structure given by

$\mathscr{C}_{\geq 0}$ = the full subcategory spanned by complexes C with $\mathrm{H}_n(C) = 0$ for all $n < 0$, and
$\mathscr{C}_{<0}$ = the full subcategory spanned by complexes C with $\mathrm{H}_n(C) = 0$ for all $n \geq 0$.

This is the *Postnikov t-structure* on $\mathcal{D}(R)$.

5.9. *Example.* For $\mathscr{C} = \mathrm{Sp}$, we have a t-structure given by

$\mathscr{C}_{\geq 0}$ = the full subcategory spanned by spectra X with $\pi_n(X) = 0$ for all $n < 0$, and
$\mathscr{C}_{<0}$ = the full subcategory spanned by spectra X with $\pi_n(X) = 0$ for all $n \geq 0$.

This is again called the *Postnikov t-structure* on Sp.

5.10. **Definition.** The *heart* of a stable ∞-category $\mathscr{C}$ with a t-structure is $\mathscr{C}^{\heartsuit} \coloneqq \mathscr{C}_{\geq 0} \cap \mathscr{C}_{\leq 0}$.

5.11. *Remark.* For $n > 0$ and objects $X, Y \in \mathscr{C}^{\heartsuit}$ we have

$$\pi_n \mathrm{Map}_{\mathscr{C}^{\heartsuit}}(X, Y) \cong \pi_n \mathrm{Map}_{\mathscr{C}}(X, Y) \cong \pi_0 \mathrm{Map}_{\mathscr{C}}(X, \Omega^n Y) = 0,$$

the last equality following since $\Omega^n Y \in \mathcal{C}_{\leq -n}$. This implies that $\mathcal{C}^\heartsuit$ is equivalent to (the nerve of) the 1-category $h\mathcal{C}^\heartsuit$. Furthermore, one can check that $h\mathcal{C}^\heartsuit$ is in fact abelian (see [BBD82, Théorème 1.3.6]): the kernel of a map $f : X \to Y$ is given by $\tau_{\geq 0}\,\mathrm{fib}(f)$ and the cokernel by $\tau_{\leq 0}\,\mathrm{cofib}(f)$. We will below not distinguish between the equivalent categories $h\mathcal{C}^\heartsuit$ and $\mathcal{C}^\heartsuit$.

5.12. **Definition.** We get a functor

$$\pi_0 : \mathcal{C} \to \mathcal{C}^\heartsuit$$

given by $\tau_{\leq 0} \circ \tau_{\geq 0}$ or equivalently $\tau_{\geq 0} \circ \tau_{\leq 0}$. We can then define

$$\pi_n : \mathcal{C} \xrightarrow{\Omega^n} \mathcal{C} \xrightarrow{\pi_0} \mathcal{C}^\heartsuit.$$

5.13. *Example.* For $\mathcal{C} = \mathcal{D}(R)$ with the Postnikov t-structure, we have

$$\mathcal{C}^\heartsuit \simeq \text{complexes with homology only in degree } 0 \simeq \mathrm{Mod}_R$$

and the functor π_n is the homology functor H_n.

5.14. *Example.* For $\mathcal{C} = \mathrm{Sp}$ with the Postnikov t-structure, we have $\mathcal{C}^\heartsuit \simeq \mathrm{Ab}$ and the functor π_n as defined above is just the “normal” definition of homotopy groups of spectra.

We now turn to the spectral sequence of a filtered object in a stable ∞-category with t-structure.

5.15. **Definition.** A *filtered object* of an ∞-category $\mathcal{C}$ is a functor $f : (\mathbb{Z}, \leq) \to \mathcal{C}$. We define

$$\mathrm{gr}_p(f) := \mathrm{cofib}(f(p-1) \to f(p)),$$

the p^{th} *graded piece* of f.

From this we obtain an exact couple

$$\begin{array}{ccc} \pi_{p+q}(f(p)) & \xrightarrow{(1,-1)} & \pi_{p+q}(f(p)) \\ & {\scriptstyle (-1,0)} \nwarrow & \downarrow {\scriptstyle (0,0)} \\ & & \pi_{p+q}(\mathrm{gr}_p(f)) \end{array}$$

where the labels on the arrows indicated the grading of the morphisms with respect to (p, q). Recall that π_i is a functor that lands in the abelian 1-category $\mathcal{C}^\heartsuit$, so we know how to do all our normal homological algebra here. The exact couple gives us a spectral sequence

$$E^1_{pq} = \pi_{p+q}(\mathrm{gr}_p\, f)$$

in $\mathcal{C}^\heartsuit$ with Serre homological differentials, *i.e.* on the r^{th} page the differentials are of the form

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

Under certain conditions, the spectral sequence converges to $\pi_{p+q}(\mathrm{colim}\, f)$: for example, if $\mathcal{C}$ has sequential colimits (*i.e.* colimits for any diagram $(\mathbb{Z}_{\geq 0}, \leq) \to \mathcal{C}$), the subcategory $\mathcal{C}_{\leq 0}$ is closed under sequential colimits, and $f(p) \simeq 0$ for $p \ll 0$. An excellent reference that discusses such issues of convergence of spectral sequences is [Boa99]; the condition mentioned in the previous sentence makes the spectral sequence a “half plane spectral sequence with exiting differentials” in Boardman’s terminology.

5.16. *Example.* If $\mathcal{C} = \mathcal{D}(R)$, then the spectral sequence that we just constructed for a filtered object of $\mathcal{D}(R)$ coincides with the classical spectral sequence for a filtered chain complex that we mentioned at the start of the section.

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