Decompositions and obstructions for the stable module ∞-category

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Abstract. This thesis consists of two research papers prefaced by an introduction. Both papers relate to the stable module ∞-category of a finite group G over a field k of characteristic p.

In the first, joint with Tobias Barthel and Jesper Grodal, we study the Picard group of the stable module ∞-category, more commonly known as the group of endotrivial modules $T(G)$. We describe the whole group of endotrivial modules in terms of p-local data, using the celebrated classification of endotrivial modules on its Sylow p-subgroup as input. In particular, we find an explicit obstruction to lifting a G-stable endotrivial S-module up to G itself, which vanishes when G has a normal p-subgroup or when $p = 2$ but does not vanish in general. As a consequence, we see that conjectures of Carlson–Mazza–Thévenaz about torsion-free endotrivial modules hold in certain cases, but not always. We illustrate the computability of our description by calculating concrete cases, such as the symmetric groups.

In the second, I show that the stable module ∞-category of a finite group G decomposes in three different ways as a limit of the stable module ∞-categories of certain subgroups of G. Analogously to Dwyer's terminology for homology decompositions, I call these the centraliser, normaliser, and subgroup decompositions. I construct centraliser and normaliser decompositions and extend the subgroup decomposition (constructed by Mathew) to more collections of subgroups. The methods used are not specific to the stable module ∞-category, so may also be applicable in other settings where an ∞-category depends functorially on G.

Resumé. Denne ph.d.-afhandling består af to forskningsartikler samt en kort indledning. Begge artikler handler om den stabile modul-∞-kategori for en endelig gruppe G over et legeme k af karakteristik p.

I den første artikel, som er skrevet i samarbejde med Tobias Barthel og Jesper Grodal, studerer vi Picard-gruppen af den stabile modul-∞-kategori, også kaldet gruppen af endotrivielle moduler $T(G)$. Vi beskriver hele gruppen $T(G)$ med p-lokal information ved at brug den fejrede klassificering af endotrivielle moduler på en p-Sylowundergruppe som input. Særligt finder vi en konkret forhindring for at løfte en G-stabil endotriviel modul til G selv, der forsvinder når G har en normal p-undergruppe eller når $p = 2$, men som ikke generelt forsvinder. Vi viser som følge af dette, at formodninger af Carlson–Mazza–Thévenaz om torsionsfri endotrivielle moduler gælder i visse tilfælde, men ikke altid. Vi illustrerer beregneligheden af vores model ved at anvende den i konkrete tilfælde, f.eks. de symmetriske grupper.

I den anden artikel viser jeg, at den stabile modul-∞-kategori kan dekomponeres på tre forskellige måder som et invers limes af de stabile modul-∞-kategorier af visse undergrupper af G. I forlængelse af Dwyers terminologi kaldes disse dekompositioner for centralisatordekompositionen, normalisatordekompositionen og undergruppedekompositionen. Jeg konstruerer centralisatordekompositionen og normalisatordekompositionen og udvider undergruppedekompositionen (først konstrueret af Mathew) til flere samlinger af undergrupper. De anvendte metoder er ikke specifikke for den stabile modul-∞-kategori, så de forventes at være anvendelige i andre situationer, hvor en ∞-kategori afhænger funktorielt af G.

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Contents

Part I

General introduction

This thesis consists of two papers:

Paper A: Torsion-free endotrivial modules via homotopy theory, and
Paper B: Decompositions of the stable module ∞-category.

The first of these is joint with Tobias Barthel and Jesper Grodal. In this general introduction, I will attempt to explain how the papers are related, with a less specialised audience in mind than the introductions to the papers themselves. To avoid too much overlap with the other introductions, I omit mention of many results from the papers, as well as background that is irrelevant to the exposition here. For more details, including a more extensive bibliography, I refer to the introductions of Papers A and B.

The common theme of the papers is the study of the stable module ∞-category using decompositions of ∞-categories. By a decomposition of an ∞-category $\mathcal{C}$, I mean an equivalence

$$\mathcal{C} \xrightarrow{\sim} \lim_{i\in I} \mathcal{C}_i$$

where I is some indexing diagram and the $\mathcal{C}_i$ are in some way simpler or easier to understand than $\mathcal{C}$. Such a decomposition allows us to study $\mathcal{C}$ by first studying each $\mathcal{C}_i$ and then attempting to "glue" our results together to say something about $\mathcal{C}$.

This is far from a new idea: the analogous concept for classifying spaces is a classical tool of homotopical group theory. Let G be a finite group and p be a prime dividing the order of G. A homology decomposition of the classifying space $\mathrm{B}G$ is a diagram of spaces $F : D \to \mathcal{S}$ such that for every $d \in D$ there is some $H \leq G$ for which $F(d) \simeq \mathrm{B}H$, together with a map

$$\operatorname{hocolim} F \to \mathrm{B}G$$

that induces an isomorphism on mod p homology. Here we have a homotopy colimit instead of a limit of ∞-categories, but the principle is the same: we can now attempt to understand G (or at least the mod p homology of its classifying space) inductively by first understanding its subgroups and then gluing this information together to study G.

Dwyer [Dwy97] identifies three types of homology decompositions, each with their own indexing category. In different ways, each of the indexing categories describes how the subgroups in $\mathscr{C}$ behave under conjugation. Let $\mathscr{C}$ be a *collection* of subgroups of G, *i.e.* a set of subgroups of G that is closed under conjugation by elements of G. Firstly, we have the *orbit category* $\mathcal{O}_{\mathscr{C}}(G)$, which is equivalent to the category whose objects are the left G-sets G/H with $H \in \mathscr{C}$ and whose morphisms are G-equivariant maps. Secondly, we have the *fusion category* $\mathcal{F}_{\mathscr{C}}(G)$, whose objects are the subgroups in $\mathscr{C}$ and whose morphisms are the group homorphisms that are induced by conjugation in G. Finally, we have the *orbit simplex category* $\bar{s}S_{\mathscr{C}}(G)$, which is the poset of the G-conjugacy classes of non-empty chains of subgroups $H_0 < \ldots < H_n$ in $\mathscr{C}$, ordered by refinement. These give us (potential) decompositions associated with $\mathscr{C}$:

Subgroup decomposition: There is a functor $F : \mathcal{O}_{\mathscr{C}}(G) \to \mathcal{S}$ with $F(G/H) \simeq \mathrm{B}H$.
Centraliser decomposition: There is a functor $F : \mathcal{F}_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}$ with $F(H) \simeq \mathrm{B}C_G(H)$.
Normaliser decomposition: There is a functor $F : \bar{s}S_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}$ with

$$F(H_0 < \ldots < H_n) \simeq \mathrm{B}\left(\bigcap_{1\leq i\leq n} N_G(H_i)\right).$$

Dwyer shows that these three are indeed decompositions of $\mathrm{B}G$ provided that $\mathscr{C}$ is an "ample" collection. There are many standard examples of ample collections, such as the collection $\mathcal{S}_p(G)$

of all non-trivial p-subgroups of G. In the cases that are of interest to us, $\mathscr{C}$ will be a subset of $\mathcal{S}_p(G)$. We let $\mathcal{O}_{\mathcal{S}}(G)$ denote the orbit category on $\mathcal{S}_p(G)$.

In my work, I am interested not in the classifying space BG, but in the stable module category of kG, where k is a field of characteristic p. Since the characteristic of k divides the order of G, we are in the realm of "modular" representation theory, and in general the module category Mod_{kG} is poorly behaved: unless the Sylow p-subgroup of G is a member of a small list of families, kG has "wild representation type", and classifying the indecomposable kG-modules would in some sense entail the simultaneous classification of the indecomposable representations of all finite dimensional k-algebras. (Specifically, the cases where G does not have wild representation type are when its Sylow is a cyclic, Klein four, dihedral, semi-dihedral, or generalised quaternion group; see Theorem 4.4.4 of [Ben98].) Such a classification is generally viewed as being unreasonably difficult. Dade [Dad78a] remarks that,

> *"There are just too many modules over p-groups! Faced with this vast, literally incomprehensible family of modules, we are reduced to looking for subfamilies which are, at the same time, small enough to be classified and large enough to be useful."*

In this spirit, the stable module category is of interest to representation theorists because it contains important information about kG-modules while still being simple enough to study.

Informally speaking, the stable module category discards the "characteristic zero" information in the module category, leaving only the behaviour that is specific to characteristic p. More precisely, to obtain the stable module category of G, we declare a morphism $f : M \to N$ in Mod_{kG} to be equivalant to the zero morphism if it factors through a projective module. We have therefore forced all projective modules to be equivalant to the zero module in the stable module category. (Recall that if the characteristic of k is zero then all modules are projective, and so the stable module category over such a field is itself zero; this is what was meant by "discarding the characteristic zero information" above.) Traditionally, the stable module category has been considered as a triangulated category, for example in Section 5 of [Car96]. However, in order to be able to get a decomposition of the stable module category, we will need to consider it as an ∞-category. In Section 2.1 of [Mat15], Mathew gives a construction of the *stable module ∞-category* StMod_{kG}, a stable ∞-category whose homotopy category is the aforementioned triangulated category. In both the triangulated and the stable ∞-category settings, the stable module (∞-)category inherits a symmetric monoidal structure from Mod_{kG}, which is given by $M \otimes_k N$ with a diagonal G-action.

It is unfortunately difficult to give anything more than a cursory explanation of the role that ∞-categories [Lur09; Lur17] play here. Informally speaking, an ∞-category behaves analogously to a category where, instead of having a unique composition of morphisms, there is only a composition that is "unique up to homotopy". An ∞-category encodes not only all the homotopies between the different compositions, but also the higher homotopies between the homotopies, and so on. This causes them in some ways to behave more like topological spaces, which is important when attempting to construct decompositions. The analogous decomposition statements for the 1-categorical stable module category are false: we have forgotten too much information by passing to the homotopy category.

Corollary 9.16 of [Mat16] proves an analogue of the subgroup decomposition for the stable module ∞-category, showing that

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{G/P \in \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP} \tag{1.1}$$

is an equivalence of symmetric monoidal ∞-categories. (Note that Mathew's result covers more collections than just $\mathcal{S}_p(G)$.) This equivalence further justifies the earlier statement that StMod_{kG} only sees the behaviour of Mod_{kG} that is specific to the prime p: each of the subgroups P appearing on the right hand side is a p-group, and $\mathcal{O}_{\mathcal{S}}(G)$ only contains data about p-subgroups of G and their normalisers (that is, the "p-local" group theory of G).

In Paper B, I prove the existence of a subgroup decomposition for more collections than those covered by Corollary 9.16 of [Mat16] and show that there are also centraliser and normaliser decompositions for many collections:

Theorem. *Let $\mathscr{C}$ be one of the collections $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{I}_p(G)$, or $\mathcal{Z}_p(G)$. There is a subgroup decomposition*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP}$$

and a normaliser decomposition

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{\bar{s}S_{\mathscr{C}}(G)} \mathrm{StMod}_{kN_G(\sigma)}.$$

If $\mathscr{C}$ is $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, or $\mathcal{Z}_p(G)$, then there is additionally a centraliser decomposition

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{\mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}_{kC_G(P)}.$$

The definitions of the collections in the statement of the theorem can all be found in the introduction to Paper B. For the subgroup decomposition, the collections that are not covered by Mathew's result are $\mathcal{B}_p(G)$, $\mathcal{I}_p(G)$, and $\mathcal{Z}_p(G)$. The proof of the theorem uses G-spaces to encode decompositions, building on the ideas in Section 3 of Dwyer's [Dwy98], and as a key input uses work of Grodal–Smith [GS06] that provides equivariant equivalences between these encoding G-spaces.

Paper A gives an example of how to use such decompositions of ∞-categories in practice. It uses Mathew's subgroup decomposition (1.1) to study the *Picard group* of the stable module ∞-category: this group consists of the equivalence classes of modules $M \in \mathrm{StMod}_{kG}$ that have an inverse under the tensor product, *i.e.* for which there exist a module $N \in \mathrm{StMod}_{kG}$ and an equivalence $M \otimes_k N \simeq k$. It has been studied by representation theorists as the *group of endotrivial modules* and is commonly denoted by $T(G)$. The name "endotrivial" arises because whenever a finite-dimensional module M has an inverse under the tensor product, the inverse must be equivalent to the k-linear dual of M; that is,

$$\mathrm{End}_k(M) \simeq M \otimes_k M^* \simeq k,$$

so the k-linear endomorphisms of M are equivalant to k as a stable G-module. Picard groups are often interesting objects to study and endotrivial modules are no exception, appearing in several places in modular representation theory.

A lot of information is already known about $T(G)$: when G is a finite p-group this group was classified in a number of important papers, starting in the 1970s with [Dad78a; Dad78b] and culminating in the early 2000s with [CT05; CT04]. For an arbitrary finite group G, we therefore want to understand the image and kernel of the restriction homomorphism

$$\mathrm{res}^G_S : T(G) \to T(S)$$

where S is a Sylow p-subgroup of G. The kernel is called the group of *Sylow-trivial* modules and was described in terms of the local group theory of G by Balmer in [Bal13] and by Grodal in [Gro18]. By earlier work of Carlson, Mazza, Nakano, and Thévenaz ([CMN06], [MT07], and [CMT13]), we also know what the image of the restriction homomorphism is as an abstract group; therefore, the aim of Paper A is to identify the image as a subgroup of $T(S)$. In the cases of interest, it turns out that $T(S)$ is a free abelian group, so we wish to identify the sublattice of $T(S)$ that corresponds to modules restricted from $T(G)$.

In order to use the subgroup decomposition to study the Picard group $T(G)$, we need to replace the Picard *group* by the Picard *space*: that is, the space of tensor-invertible objects in StMod_{kG} together with the equivalences between them, and homotopies between those equivalences, and the homotopies between those homotopies, and so on. This is a Kan complex whose set of connected components is (by definition) isomorphic to the Picard group. A result of Mathew–Stojanoska [MS16, Proposition 2.2.3] tells us that the subgroup decomposition induces a decomposition of the Picard space as a homotopy limit:

$$\mathrm{Pic}(\mathrm{StMod}_{kG}) \xrightarrow{\sim} \operatorname*{holim}_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} \mathrm{Pic}(\mathrm{StMod}_{kP}).$$

Our algebraic question has become topological, and standard techniques from algebraic topology give an obstruction that determines the image of the restriction:

Theorem. *There is an exact sequence*

$$0 \to \mathrm{H}^1(\mathcal{O}_{\mathcal{S}}(G); k^\times) \to T(G) \to T^{\,G\text{-stable}}(S) \xrightarrow{\alpha} \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times),$$

and we can explicitly describe a 2-cocycle representing the obstruction α.

Here $T^{\,G\text{-stable}}(S)$ denotes the group of *G-stable endotrivial S-modules*, which is the subgroup of $T(S)$ that is invariant under G-conjugation. It is isomorphic to $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$, which is how we denote it throughout Paper A, and it is straightforward to show that it contains the image of the restriction morphism res^G_S.

The rest of Paper A involves manipulating the obstruction α in order to obtain an algebraic model for $T(G)$, expressed in terms of the p-subgroups of G and the one-dimensional characters of their normalisers. Since the groups of characters in question have often already been computed, this model is useful for calculations. Along the way, we provide an algebraic criterion for showing that a given G-stable endotrivial S-module lifts to $T(G)$: a module lifts if and only if it is possible to specify a one-dimensional character of $N_G(P)$ for each p-subgroup $P \leq G$ such that these characters satisfy a compatibility condition on the cyclic p-subgroups. We carry out these calculations for the group $G = \mathrm{PSL}_3(p)$, where $p \equiv 1 \pmod 3$, showing that the image of the restriction is a subgroup of $T(S)$ of index three. This allows us to settle two conjectures of Carlson–Mazza–Thévenaz [CMT14, Conjectures 9.2 and 10.1], partially positively and partially negatively.

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Part II

Torsion-free endotrivial modules via homotopy theory

TORSION-FREE ENDOTRIVIAL MODULES VIA HOMOTOPY THEORY

TOBIAS BARTHEL, JESPER GRODAL, AND JOSHUA HUNT

Abstract. The group of endotrivial kG-modules $T(G)$, for G a finite group and k a field of characteristic p, is by definition the Picard group of the stable module category of kG-modules. The quest to understand this group has a long history, as it occurs in many parts of representation theory. In this paper, we describe the whole group of endotrivial modules of an arbitrary finite group in terms of computable p-local data, using the celebrated classification of endotrivial modules on its Sylow p-subgroup S as input. We accomplish this by combining techniques from higher algebra with recent homotopy methods of the second-named author, which he used to describe the subgroup of torsion endotrivial modules. In particular, we find an explicit obstruction to lifting a G-stable endotrivial S-module up to G itself, which vanishes when G has a normal p-subgroup or when $p = 2$ but does not vanish in general. As a consequence, we see that conjectures of Carlson–Mazza–Thévenaz about torsion-free endotrivial modules hold in certain cases, but not always. We illustrate the computability of our description by calculating concrete cases, such as the symmetric groups.

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1. Introduction

Endotrivial modules, *i.e.* kG-modules such that $\mathrm{End}_k(M) \cong k \oplus (\mathrm{proj})$, for G a finite group and k a field of characteristic p, play an important role in many parts of representation theory, which is surveyed in [Thé07; Car12; Car17]. They form an abelian group under tensor product, denoted $T(G)$ or $T_k(G)$. This is the group of tensor-invertible objects in the stable module category of G, and it is known to be finitely generated [CMN06, Corollary 2.5].

When P is a finite p-group, $T(P)$ was classified in a number of important papers, starting in the 1970s with [Dad78a; Dad78b] and culminating in the early 2000s with [CT05; CT04]. For G a general finite group, computable formulas for the torsion part of $T(G)$ in terms of p-local group theory were provided recently by the second-named author in [Gro18]. The goal of this paper is to combine the homotopy methods from [Gro18] with additional tools from higher algebra to obtain a computable model for the whole of $T(G)$. In particular, we obtain information about the torsion-free generators, settling conjectures of Carlson–Mazza–Thévenaz [CMT14] both in the positive and in the negative.

We briefly recall the outline of the classification of $T(P)$; more details on the known structure of $T(G)$ are provided in Section 2.6. Dade showed in [Dad78a; Dad78b] that when P is a finite abelian p-group, $T(P)$ consists of shifts of the trivial module, *i.e.* it is infinite cyclic unless P itself is cyclic, in which case it is 0 if $|P| \leq 2$ and $\mathbb{Z}/2$ otherwise. Carlson–Thévenaz showed in [CT05] that when P is an arbitrary finite p-group, $T(P)$ is torsion-free except in the exceptional cases when the p-group is cyclic, a semi-dihedral 2-group, or a generalised quaternion 2-group. In all the exceptional cases, we know explicit generators for $T(P)$. In [CT04], they also established generators in the cases where $T(P)$ is torsion-free, proving that a set of elements given earlier by Alperin [Alp01] that was known to form a rational basis is in fact an integral basis.

1.1. Obstruction theory.

Since we understand the group of endotrivial modules for p-groups, we can aim to understand $T(G)$ for arbitrary finite G by choosing a Sylow p-subgroup S of G and considering the exact sequence

$$0 \to T(G,S) \to T(G) \xrightarrow{\mathrm{res}} T(S). \tag{1.2}$$

Here $T(G,S)$ is defined as the kernel of the restriction map, *i.e.* the modules M such that $\mathrm{res}^G_S M \cong k \oplus (\mathrm{proj})$, referred to as *Sylow-trivial* modules. In [Gro18, Theorem A], the second-named author established an isomorphism $T(G,S) \cong \mathrm{H}^1(\mathcal{O}_{\mathcal{S}}(G); k^\times)$, building on earlier ideas of Balmer [Bal13]. Here $\mathcal{O}_{\mathcal{S}}(G)$ denotes the orbit category of G on the collection $\mathcal{S}_p(G)$ of non-trivial p-subgroups of G. This cohomological description is very amenable to manipulation via standard methods from homotopy theory, and [Gro18, Theorem D] says that $T(G,S)$ can alternatively be described as a collection of one-dimensional characters of normalisers of chains $N_G(P_0 < \dots < P_n)$, compatible under refinement. This implies the Carlson–Thévenaz conjecture [Gro18, Theorem F], which gives an algorithmic way of calculating $T(G,S)$ from p-local group theory information. In terms of homotopy theory, the statement about one-dimensional characters can again be rewritten as $T(G,S) \cong \mathrm{H}^0_G(\mathcal{S}_p(G); \mathrm{H}^1(-; k^\times))$, where here $\mathrm{H}^*_G(X; F)$ denotes the G-equivariant Bredon cohomology of a G-space X with coefficients in the functor F; see [Gro18, §5].

Understanding the exact image of the restriction $T(G) \to T(S)$ is subtle, as we now explain. Carlson–Mazza–Nakano [CMN06, Theorem 3.1] show that the rank of $TF(G)$ can be described by a formula similar to Alperin's formula for p-groups [Alp01, Theorem 4]:

$$\dim_{\mathbb{Q}}(T(G)\otimes\mathbb{Q}) = \mathrm{rk}(TF(G)) = \begin{cases} 0 & \text{if } \mathrm{rk}_p(G)=1,\\ n_G & \text{if } \mathrm{rk}_p(G)=2, \text{ and}\\ n_G+1 & \text{if } \mathrm{rk}_p(G)\geq 3.\end{cases} \tag{1.3}$$

Here $\mathrm{rk}_p(G)$ is the p-rank of G, *i.e.* the rank r of the largest elementary abelian p-subgroup $(\mathbb{Z}/p)^r$ of G, and n_G is the number of conjugacy classes of rank two maximal elementary abelian subgroups of G. This formula determines $TF(G)$ as an abstract group. Apart from the group theoretic question of actually calculating n_G for classes of groups (see Section 2.6), the above rank formula for $TF(G)$ also still leaves open the important question of describing torsion-free generators for $T(G)$, for example via their image in $T(S)$, and pinning down its representation theoretic structure—a status in some ways analogous to the status in the p-group case prior to [CT04]. This has been the subject of a number of works, both in the abstract and for concrete collections of finite groups; the paper [CMT14] summarises many of the pre-existing approaches to this problem.

An obvious condition on the image of the restriction $T(G)\to T(S)$ is that it must land in the subgroup $\lim_{\mathcal{O}_{\mathcal{S}}^{\mathrm{op}}} T(-)$ of *G-stable* modules in $T(S)$, *i.e.* the endotrivial S-modules that restrict to compatible elements under restriction and conjugation in G. Using methods from higher algebra we find a computable obstruction for this being the case. Extending the results in [Gro18] "to the right" we get the following sequences describing $T(G)$ in terms of $T(S)$ and p-local information.

Theorem A. *There are exact sequences*

$$0\to \mathrm{H}^1(\mathcal{O}_{\mathcal{S}}(G);k^\times)\to T(G)\xrightarrow{\mathrm{res}} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)\xrightarrow{\alpha}\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);k^\times)$$

and

$$0\to \mathrm{H}^0_G(\mathscr{C};\mathrm{H}^1(-;k^\times))\to T(G)\xrightarrow{\mathrm{res}} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)\xrightarrow{\beta}\mathrm{H}^1_G(\mathscr{C};\mathrm{H}^1(-;k^\times)),$$

where $\mathscr{C}$ is any collection of non-trivial p-subgroups of G for which $\mathscr{C}\hookrightarrow\mathcal{S}_p(G)$ is a G-homotopy equivalence (Theorem 4.5 and Corollary 5.10). We can explicitly describe cocycles representing the obstructions α and β (Theorem 4.5 and Proposition 5.11).

The two left-most terms in the exact sequences are descriptions of $T(G,S)$ from [Gro18], as explained above. All of the cohomology groups mentioned are finite abelian p'-groups described in terms of the p-local structure of G. We provide information on cases when α and β are known either to vanish or to be non-zero in Theorem C; in particular, when $p=2$ the obstruction classes vanish, but at odd primes they do not vanish in general.

Previously, Balmer [Bal15] provided an obstruction to lifting modules from a subgroup H (of p'-index) to G, lying in a Čech cohomology group with respect to a "sipp topology" that he introduced. While obviously fundamental and an inspiration for this work, the theory was not immediately amenable to calculations of the obstruction group or obstruction class. In Appendix A—logically independent of the rest of the paper—we go back and show that, when H is a Sylow p-subgroup, Balmer's obstruction group is in fact isomorphic to $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}};k^\times)$ and his obstruction corresponds to α under our constructed isomorphism.

1.4. **An algebraic model.** As mentioned previously, the obstruction class in Theorem A may be non-zero when p is odd. Just as an element of the cohomology group $\mathrm{H}^0_G(\mathscr{C};\mathrm{H}^1(-;k^\times))$ can be thought of as a set of one-dimensional characters $N_G(P)\to k^\times$ that agree as P varies, we can view the obstruction group $\mathrm{H}^1_G(\mathscr{C};\mathrm{H}^1(-;k^\times))$ as encoding the obstruction to the existence of such a set of characters. It therefore makes sense to build these characters into a model for $T(G)$, which is the content of our Theorem B.

Given an endotrivial module $M \in T(G)$, the most obvious invariant that we can assign to it is the data of its restrictions to elementary abelian p-subgroups. We let $\mathcal{A}_p(G)$ denote the poset of non-trivial elementary abelian p-subgroups of G. Recall that Dade's theorem implies that, for $V \in \mathcal{A}_p(G)$, every endotrivial V-module is a shift of the trivial module. We can therefore choose a *type function* $n : \mathcal{A}_p(G) \to \mathbb{Z}$ that satisfies

$$\operatorname{res}^G_V M \simeq \Omega^{n(V)} k \tag{1.5}$$

for every $V \in \mathcal{A}_p(G)$. This function determines the image of M in $TF(G)$ but not M itself [CMT14, Theorem 2.2].

Having chosen such a type function, we can assign further invariants to M, which describe the action of p'-elements of G: for any $V \in \mathcal{A}_p(G)$, we consider the $N_G(V)$-module $\hat{\mathrm{H}}^{n(V)}(V; M)$, which is one-dimensional because of (1.5). We let $\varphi_V \in \mathrm{H}^1(N_G(V); k^\times)$ denote the one-dimensional character of $N_G(V)$ that corresponds to the aforementioned module.

These invariants satisfy compatibility conditions because they come from a G-module. Since the restriction of M to $T(S)$ is an endotrivial S-module, the type function n must satisfy conditions imposed by the classification of endotrivial modules on $T(S)$, as in [CMT14, §3]. Furthermore, since the restriction of M to $T(S)$ is G-stable, the type function must be constant on G-conjugacy classes of elementary abelian subgroups.

The compatibility conditions on the one-dimensional characters relate to the action of p'-elements on cyclic subgroups: for a cyclic p-subgroup $V \leq S$, the fact that $T(V) \cong \mathbb{Z}/2$ imposes a compatibility condition on the type function n and hence on the possible structures of a module in $T(S)$. However, when we lift to G, we might have a p'-element x that acts non-trivially on V, in which case the order of Ωk in $T(V \rtimes \langle x \rangle)$ is greater than two. The compatibility condition on the one-dimensional characters arises from this larger periodicity, and this phenomenon is also exactly what our obstructions in Theorem A are measuring. We say that a set of characters satisfying this compatibility condition is an *orientation* for n; see Definition 7.6. We work through the concrete example $G = \mathrm{PSL}_3(p)$ in Example 10.2, where this phenomenon is clearly visible; that example does not use any of the obstruction theory that we develop in the first part of the paper, and some readers may find it useful to skip ahead to the example before reading the intervening sections.

Our algebraic model for $T(G)$ says that the compatibility conditions in the above two paragraphs determine a unique endotrivial G-module:

Theorem B. *The group $T(G)$ is isomorphic to the abelian group of tuples $(M_S, n : \mathcal{A}_p(G) \to \mathbb{Z}, \{\varphi_V : V \in \mathcal{A}_p(G)\})$, where M_S is a G-stable endotrivial S-module, n is a type function for M_S, and $\{\varphi_V\}$ is an orientation for n, under an explicit equivalence relation (Theorem 8.3).*

There is also an isomorphism

$$T(G) \cong \lim_{[P_0 < \ldots < P_n]} T(N_G(P_0 < \ldots < P_n)),$$

where the indexing of the limit is over the poset of conjugacy classes of non-empty chains of p-subgroups of G, ordered by refinement (Theorem 9.4). In particular, $T(G)$ is an invariant of the p-local structure of G.

The conditions for being a type function and an orientation, as well as the equivalence relation, are all straightforward to verify for explicit tuples, so we obtain a computable model for $T(G)$. The description of $T(G)$ in terms of the endotrivial modules of normalisers of chains of subgroups $T(N_G(P_0 < \ldots < P_n))$ generalises the normaliser decomposition of [Gro18, Theorem D].

1.6. **(Non)-vanishing of the obstructions.** We study the behaviour of the obstructions α and β appearing in Theorem A using a range of techniques from homotopy theory. Here we collect our results on when they vanish or not:

Theorem C. *If $p = 2$ or G has a non-trivial normal p-subgroup, then α and β are zero (Corollary 7.12 and Proposition 5.8); in this case, we obtain an exact sequence*

$$0 \to T(G,S) \to T(G) \xrightarrow{\text{res}} \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-) \to 0.$$

If α or β is zero and p is odd, then we obtain that

$$TF(G) \xrightarrow{\cong} \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} TF(-)$$

via restriction (Proposition 11.1), where the right-hand side denotes the G-stable elements in $TF(S)$.

In general, α and β are non-zero: for example, when $G = \mathrm{PSL}_3(p)$ and $p \equiv 1 \bmod 3$, we have that $\mathrm{H}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times)) \cong \mathbb{Z}/3$ and β is surjective (Example 10.2), so we obtain an exact sequence

$$0 \to T(G) \to \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-) \to \mathbb{Z}/3 \to 0.$$

Even though our theorem says that our obstructions may be non-zero in general when p is odd, the cases where this occurs seem to be very limited, for a range of reasons. It may be possible to classify exactly when this happens for many classes of groups, for example the simple groups, by using our model from Theorem B in combination with the well-developed theory for calculating the obstruction groups that is demonstrated in [Gro18]. As an example, we carry this out for symmetric groups in Section 12: the image of the restriction is a proper subgroup of $\lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-)$, and we show that a a rational basis constructed by Carlson–Hemmer–Mazza in [CHM10] is also an integral basis.

Theorem C sheds light on some open questions in the literature about p-fusion invariance of $TF(G)$ and choice of generators:

Corollary D. *Suppose that $\phi : G \to G'$ is a group homomorphism that controls p-fusion. The induced homomorphism $\phi^* : TF(G') \to TF(G)$ is an isomorphism if $p = 2$. If p is odd, this is not necessarily the case; for example, when $p \equiv 1 \bmod 3$ the quotient map $\phi: \mathrm{SL}_3(p) \to \mathrm{PSL}_3(p)$ preserves fusion but the induced map $\phi^* : TF(\mathrm{PSL}_3(p)) \hookrightarrow TF(\mathrm{SL}_3(p))$ is the inclusion of a subgroup of index three.*

This answers [CMT14, Conjecture 10.1] in the positive when $p = 2$, but in the negative in general; it is proved as Theorem 11.3.

A related question is what qualitative properties the generators of $TF(G)$ possess: the previously known methods for building generators have all involved constructing them from shifts of the trivial module, which led to the conjecture [CMT14, Conjecture 9.2] that these might all be chosen to lie in the principal block of kG, *i.e.* the block containing the trivial module. As not all endotrivial modules for $\mathrm{SL}_3(p)$ are inflated from $\mathrm{PSL}_3(p)$, this conjecture was too optimistic:

Corollary E. *When $p \equiv 1 \bmod 3$, a generating set of $TF(\mathrm{SL}_3(p))$ cannot be chosen in the principal block.*

This is proved as Theorem 11.4.

1.7. **Outline of proofs.** Let us now sketch how we prove these theorems, and give an outline of the paper. The key starting point for our approach is to combine the viewpoint in [Gro18] with the power of ∞-categories, which enables us to set up the obstruction theory for lifting a compatible family of endotrivial kP-modules, for P ranging over p-subgroups of G, to an endotrivial kG-module. More precisely, using a result of Mathew [Mat16], we obtain that

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \operatorname*{holim}_{G/P \in \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP}$$

as symmetric monoidal ∞-categories (see Proposition 3.1). The Picard space $\mathrm{Pic}(-)$ is a functor from symmetric monoidal ∞-categories to pointed spaces, which commutes with homotopy limits [MS16]. Applying this functor produces a homotopy decomposition of *spaces*

$$\mathrm{Pic}(\mathrm{StMod}_{kG}) \xrightarrow{\sim} \operatorname*{holim}_{G/P \in \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} \mathrm{Pic}(\mathrm{StMod}_{kP}).$$

Once in the world of spaces, there is a classical spectral sequence and obstruction theory, developed by Bousfield–Kan [BK72] and others, for calculating the homotopy groups of a homotopy limit of a diagram $F: I \to \mathcal{S}$ of spaces in terms of the derived functors of the inverse limit of the homotopy groups of the spaces $F(i)$. The spectral sequence has the slight complication that it is "fringed", which is related to problems with strictifying basepoints; indeed, determining whether or not a basepoint lifts is our main concern! However, spectral sequences of this form have been intensively studied in unstable homotopy theory, for example in connection with calculating maps between classifying spaces, and there is also an extensive literature on calculating the derived limits that appear (see *e.g.* [Oli98; Gro02; Gro10; Gro18]). As the homotopy groups of the Picard space $\mathrm{Pic}(\mathrm{StMod}_{kP})$ identify in positive degrees with Tate cohomology, we can use a vanishing result of Jackowski–McClure for derived limits with values in p-local Mackey functors to show that only one "existence" and one "uniqueness" group might be nontrivial, arriving at the first exact sequence in Theorem A. This is done in Section 3 and Section 4, with preliminaries in Section 2. We then use the isotropy spectral sequence of a collection $\mathscr{C}$ to relate the obstruction class $\alpha \in \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$ to Bredon cohomology, as in [Gro18], and derive the second exact sequence in Theorem A.

To deduce Theorem B and Theorem C from this, we have to analyse the obstruction class $\beta \in \mathrm{H}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times))$, and understand it in terms of representation theory. As alluded to prior to the statement of Theorem B, the key point is that for a kG-module M, the normaliser $N_G(P)$ acts on $\mathrm{res}^S_P M$ in the homotopy category of StMod_{kP}, giving rise to a one-dimensional character of $N_G(P)$ when P is abelian. Since the restrictions of M to non-conjugate abelian p-subgroups might be equivalent to different shifts of the trivial module, we obtain a relation between the one-dimensional characters on these subgroups that is more complicated than the relation in the Sylow-trivial case in [Gro18, Theorem D]. For a G-stable endotrivial S-module, the obstruction in $\mathrm{H}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times))$ vanishes if these relations can be simultaneously satisfied.

In Section 6, we carry out an analysis of actions on objects in the homotopy category of StMod_{kP}, as in the above paragraph, and use it in Section 7 to reinterpret β in terms of type functions and orientations. We thereby see that the obstruction to lifting a G-stable endotrivial S-module can be seen directly, via representation theory, without using the cohomological obstruction theory. This reinterpretation is used in Sections 8 and 9 to prove the two parts of Theorem B. The remaining sections give other consequences and computations, and the appendix establishes the relationship between the obstruction class α and Balmer's work.

We end this discussion of proofs by pointing out that the method of proof is rather general, combining ∞-categorical techniques with algebraic obstruction group calculations as they occur in classical homotopy theory, and hence should apply to other to other symmetric monoidal ∞-categories in algebra and topology.

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2. NOTATION AND PRELIMINARIES

Throughout the paper, we let G be a finite group, p be a prime dividing its order, k be a field of characteristic p, and S be Sylow p-subgroup of G.

We use the unadorned word "space" to mean "simplicial set". Since we will need to work with both 1-categorical and ∞-categorical (co)limits, we will call the former "(co)limits" and the latter "homotopy (co)limits".

2.1. **Group theory.** Let $\mathcal{S}_p(G)$ denote the poset of non-trivial p-subgroups of G, let $\mathcal{A}_p(G)$ denote its subposet of non-trivial elementary abelian p-subgroups, and let $\mathcal{B}_p(G)$ denote the poset of non-trivial p-radical subgroups of G (*i.e.* subgroups $P \in \mathcal{S}_p(G)$ with $P = O_p(N_G(P))$, where $O_p(H)$ is the largest normal p-subgroup of H).

By a *collection* of subgroups we mean a set $\mathscr{C}$ of subgroups of G that is closed under conjugation: for example, the posets $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, and $\mathcal{S}_p(G)$ introduced in the previous paragraph. For any such collection $\mathscr{C}$, we write $\mathcal{T}_{\mathscr{C}}(G)$ for its *transport category, i.e.* the Grothendieck construction of the left action of G on $\mathscr{C}$. More explicitly, $\mathcal{T}_{\mathscr{C}}(G)$ is the category whose objects are the elements of $\mathscr{C}$ and whose morphisms are given by

$$\mathrm{Mor}(P,Q) := \{\, g \in G \,:\, {}^gP \le Q \,\},$$

where ${}^gP$ denotes gPg^{-1}. When referring to the transport category of a named collection such as $\mathcal{S}_p(G)$, we will drop the decorations and write $\mathcal{T}_{\mathcal{S}}(G)$ (similarly for the categories $\mathcal{O}_{\mathcal{S}}(G)$ and $\mathcal{F}_{\mathcal{S}}(G)$ defined below).

We have an *orbit category* $\mathcal{O}_{\mathscr{C}}(G)$ associated with such a collection, namely the category of transitive G-sets with isotropy groups in $\mathscr{C}$ and G-equivariant maps between them. We can again be more explicit: this category is equivalent to the category with objects G/P for $P \in \mathscr{C}$ and morphisms given by

$$\mathrm{Mor}(G/P, G/Q) := \{\, g \in G \,:\, P^g \le Q \,\}/Q,$$

where P^g denotes $g^{-1}Pg$. There is a canonical quotient functor $\mathcal{T}_{\mathscr{C}}(G) \to \mathcal{O}_{\mathscr{C}}(G)$ that sends P to G/P and sends a morphism $g \in \mathrm{Mor}(P,Q)$ to the coset $g^{-1}Q \in \mathrm{Mor}(G/P, G/Q)$. We will sometimes need to refer to the orbit category on *all* p-subgroups, not just the non-trivial p-subgroups, and we will denote this by $\mathcal{O}_p(G)$. This is the same as adding the free G-set to $\mathcal{O}_{\mathcal{S}}(G)$.

We also have a *fusion category* $\mathcal{F}_{\mathscr{C}}(G)$ associated with such a collection. The objects of $\mathcal{F}_{\mathscr{C}}(G)$ are the elements of $\mathscr{C}$ and the morphisms are the group homomorphisms that are induced by conjugation in G:

$$\mathrm{Mor}(P,Q) \cong \{\, g \in G \,:\, {}^gP \le Q \,\}/C_G(P).$$

There is again a natural quotient functor $\mathcal{T}_{\mathscr{C}}(G) \to \mathcal{F}_{\mathscr{C}}(G)$ that is the identity on objects and the natural quotient on morphism sets.

Finally, we have an *orbit simplex category* $\overline{s}S_{\mathscr{C}}(G)$, which is the poset of G-conjugacy classes of chains of subgroups in $\mathscr{C}$: we let $\overline{\sigma} \le \overline{\tau}$ if σ is a face of τ for some representatives $\sigma \in \overline{\sigma}$ and $\tau \in \overline{\tau}$. (This is the opposite of the category with this name in [Dwy98, §1.7].) This category is equivalent to the category whose objects are chains of subgroups in $\mathscr{C}$, and where there is a unique morphism $\sigma \to \tau$ when there is an element of G conjugating σ to a face of τ.

2.2. **The stable module ∞-category of a finite group.** We write Mod_{kG} for the category of kG-modules, with the symmetric monoidal structure $(\otimes_k, k)$ coming from the Hopf algebra structure of kG. A map $f\colon M \to N$ is called a *stable equivalence* if there exists $g\colon N \to M$ such that $fg - \mathrm{id}_N$ and $gf - \mathrm{id}_M$ both factor through a projective kG-module. Inverting the stable equivalences on Mod_{kG} yields the *stable module category*, which is a triangulated category that has been studied by representation theorists, for example as a way of classifying kG-modules when Mod_{kG} has wild representation type; see [BIK11].

In order to study decompositions of the stable module category over the orbit category of G, we need to work within a suitable higher categorical setting [Lur09; Lur17]. Therefore, we consider primarily the *stable module ∞-category* StMod_{kG}, which is defined as the ∞-categorical localisation of Mod_{kG} at the stable equivalences. We will use $\underline{\mathrm{Hom}}_G(M,N)$ to denote the mapping space between objects $M, N \in \mathrm{StMod}_{kG}$. The homotopy category of StMod_{kG} is the stable module category as defined in the previous paragraph. Its objects are kG-modules and its hom sets are given by

$$\pi_0 \underline{\mathrm{Hom}}_G(M,N) \cong \mathrm{Hom}_G(M,N)/(f \sim 0 \text{ if } f \text{ factors through a projective}).$$

Every stable ∞-category $\mathcal{C}$ comes equipped with a *shift* or *desuspension* functor $\Omega\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$. (See [Lur17, §1] for a general introduction to the theory of stable ∞-categories.) For $\mathcal{C} = \mathrm{StMod}_{kG}$, this identifies with the *Heller shift*, which can be computed by taking the kernel of a projective cover. The salient features of StMod_{kG} are summarised in the following theorem:

2.3. **Theorem.** *The stable module ∞-category* StMod_{kG} *has the structure of a symmetric monoidal stable ∞-category with unit k, where the monoidal structure is inherited from the symmetric monoidal structure on* Mod_{kG}. *Moreover,* StMod_{kG} *is rigidly compactly generated,* i.e. *it admits a set of compact generators and the compact objects coincide with the dualisable ones.*

See [Car96, §5] for a discussion of the triangulated category and [Mat15, Definition 2.2] for a construction of StMod_{kG} as a symmetric monoidal stable ∞-category.

2.4. *Remark.* More explicitly, the stable module ∞-category may be constructed as the underlying ∞-category of a model structure on Mod_{kG} for which:

- weak equivalences are given by stable equivalences,
- cofibrations are precisely the monomorphisms, and
- fibrations are precisely the epimorphisms in Mod_{kG}.

We refer to [Hov99, §2.2] for a more detailed discussion of this model structure.

Given a subgroup $H \to G$ and an element $g \in G$, restricting along the right-conjugation map $c_g\colon {}^gH \to H$ gives a functor

$$g \otimes_H -\colon \mathrm{StMod}_{kH} \to \mathrm{StMod}_{k({}^gH)}.$$

Explicitly, $g \otimes_H M$ has the twisted action $ghg^{-1} \cdot (g \otimes_H m) := g \otimes_H (h \cdot m)$. Note that $g \otimes_H M$ can be identified with a submodule of the induced module $kG \otimes_H M$. More generally, restriction to a subgroup $i\colon H \to G$ gives rise to a symmetric monoidal functor

$$\mathrm{res}^G_H = i^*\colon \mathrm{StMod}_{kG} \to \mathrm{StMod}_{kH}$$

which preserves all (homotopy) limits and (homotopy) colimits. This implies that i^* admits a left adjoint $i_!$ and a right adjoint i_*, usually referred to as induction and coinduction, respectively. Since G/H is finite, there is in fact a natural equivalence $i_* \simeq i_!$. Since i_* inherits a lax monoidal structure from i^*, it follows that i_*k is a commutative algebra object in StMod_{kG}, whose underlying kG-module can be identified with $\prod_{G/H} k$ with its permutation action, and whose multiplication is given by pulling back along the diagonal map $G/H \to G/H \times G/H$. We will denote this algebra object by A_H^G. Balmer [Bal15, Theorem 1.2] proves an equivalence between StMod_{kH} and the category of modules over A_H^G internal to StMod_{kG}; see [Mat16, Proposition 9.12] for the ∞-categorical version of this theorem.

2.5. **Theorem** (Balmer, Mathew). *For any subgroup $H \leq G$, there is a natural symmetric monoidal equivalence*

$$\mathrm{StMod}_{kH} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A_H^G),$$

induced by coinduction, under which the free/forget adjunction $\mathrm{StMod}_{kG} \leftrightarrows \mathrm{Mod}_{\mathrm{StMod}_{kH}}(A_H^G)$ *corresponds to the restriction/coinduction adjunction* $\mathrm{StMod}_{kG} \leftrightarrows \mathrm{StMod}_{kH}$.

We can extend the construction of the stable module ∞-category to an arbitrary finite G-set X by setting

$$\mathrm{StMod}(X) := \mathrm{Mod}_{\mathrm{StMod}_{kG}}\left(\textstyle\prod_X k\right),$$

where the algebra structure on $\prod_X k$ is again given by pullback along the diagonal map $X \to X \times X$. (More formally, this is a homotopy Kan extension of StMod_{k-} along the inclusion of the orbit category into all finite G-sets.) With this notation, Theorem 2.5 provides an equivalence $\mathrm{StMod}(\coprod_{i\in I} G/H_i) \simeq \prod_{i\in I} \mathrm{StMod}_{kH_i}$ for any finite set of transitive G-sets $\{G/H_i \, : \, i \in I\}$.

2.6. **Endotrivial modules.** Our primary object of study is the group of endotrivial modules of G, denoted $T(G)$. A kG-module M is said to be *endotrivial* if

$$M^* \otimes_k M \simeq k \oplus (\mathrm{proj})$$

as kG-modules, where M^* is the k-linear dual of M and (proj) denotes some projective kG-module. In particular, a finitely generated module M is endotrivial if and only if its k-endomorphism ring is stably equivalent to the trivial module. We write $[M]$ for the stable equivalence class of M in StMod_{kG}, and define

$$T(G) := \{\, [M] \, : \, M \text{ endotrivial} \,\},$$

which is an abelian group under the operation $[M] + [N] := [M \otimes_k N]$. (We will henceforth omit the subscript k in $\otimes_k$.) The inverse of $[M]$ is its k-linear dual $[M^*]$. For any finite group G, the abelian group $T(G)$ is finitely generated: see [CMN06, Corollary 2.5] and earlier work of Puig [Pui90, Corollary 2.4] for the case of finite p-groups.

A fact about endotrivial modules that we will use repeatedly is that their endomorphisms are all homotopic to multiplication by a scalar:

2.7. **Lemma.** *If M is an endotrivial G-module, then $\pi_0 \, \underline{\mathrm{End}}_G(M) \cong k$ as k-algebras.*

Proof. Since M is necessarily a compact object of StMod_{kG}, we have

$$\underline{\mathrm{Hom}}_G(M, M) \simeq \underline{\mathrm{Hom}}_G(k, M^* \otimes M) \simeq \underline{\mathrm{Hom}}_G(k, k),$$

so up to homotopy any endomorphism of M is given by $M \otimes \varphi$ for some $\varphi \in \underline{\mathrm{Hom}}_G(k, k)$. □

One key property of the shift functor Ω introduced previously is that $\Omega^n N \otimes \Omega^m M \simeq \Omega^{m+n}(N \otimes M)$. In particular, for every finite group G, the shifts $\{\, \Omega^n k \, : \, n \in \mathbb{Z} \,\}$ of the trivial module form a cyclic subgroup of $T(G)$ generated by Ωk. A classical theorem of Dade says that for abelian p-groups, this is the entire group of endotrivial modules:

2.8. **Theorem** ([Dad78b, Proposition 9.16, Theorem 10.1]). *If P is a non-trivial abelian p-group then $T(P)$ is cyclic, generated by Ωk_P. Furthermore:*

$$T(P) \cong \begin{cases} 0 & \text{if } P \cong \mathbb{Z}/2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } P \cong \mathbb{Z}/n \text{ with } n > 2, \text{ and} \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

A lot is already known about the structure of $T(G)$. In addition to Dade's theorem above, Carlson–Thévenaz showed in [CT05] that when P is an arbitrary finite p-group, $T(P)$ is torsion-free except in the cases where the p-group is cyclic, a semi-dihedral 2-group (where the torsion subgroup is $\mathbb{Z}/2$), or a generalised quaternion 2-group (where it is either $\mathbb{Z}/4$ or $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, depending on whether or not k has a primitive third root of unity).

Carlson–Mazza–Nakano [CMN06, Theorem 3.1] explicitly describe the torsion-free rank of $T(G)$:

$$\dim_{\mathbb{Q}}(T(G) \otimes \mathbb{Q}) = \mathrm{rk}(TF(G)) = \begin{cases} 0 & \text{if } \mathrm{rk}_p(G) = 1, \\ n_G & \text{if } \mathrm{rk}_p(G) = 2, \text{ and} \\ n_G + 1 & \text{if } \mathrm{rk}_p(G) \geq 3, \end{cases} \tag{2.9}$$

where $\mathrm{rk}_p(G)$ is the p-rank of G, *i.e.* the rank r of the largest elementary abelian p-subgroup $(\mathbb{Z}/p)^r$ of G, and n_G is the number of conjugacy classes of rank two maximal elementary abelian p-subgroups of G. This generalises Alperin's formula for p-groups [Alp01, Theorem 4].

The number n_G is related to many questions in classical group theory, and there has been a lot of research into determining its value; see *e.g.* [Mac70; GM10; CMN06; CGMN20]. In particular, unless G itself has low rank, the number n_G is zero and the torsion free part of $T(G)$ is of rank one, generated by shifts of the trivial module. Specifically, by "low rank" we mean less than or equal to p if p is odd [GM10, Theorem A] and less than or equal to four if $p = 2$, by [Mac70, "Four generator theorem"] together with [GM10, Lemma 2.4]. The origin of the formula in (2.9) is that it computes the number of connected components of the poset of conjugacy classes of elementary abelian p-subgroups of G of rank at least two; on these subgroups the torsion-free rank is known to be one by Dade's theorem. (We sketch a reproof of (2.9) in Remark 4.8, using the obstruction theory of Section 4.)

In light of (2.9), to determine $T(G)$ as an abstract group we only need to calculate its torsion subgroup. Since $T(S)$ is torsion-free except in the few exceptional cases listed above, this is in most cases equal to the kernel $T(G, S)$ of the restriction map $T(G) \to T(S)$, also known as the group of *Sylow trivial modules*. More explicitly, these are modules $M \in T(G)$ such that $\mathrm{res}^G_S M \cong k \oplus (\text{free})$. In [Gro18], the second-named author established several descriptions of $T(G, S)$ in terms of local group theory, building on earlier ideas of Balmer [Bal13]. Here we list the descriptions that we will use later:

- There is an isomorphism $T(G, S) \cong \mathrm{H}^1(\mathcal{O}_{\mathcal{S}}(G); k^\times)$. [Gro18, Theorem A]
- Let $\mathscr{C}$ be a collection of p-subgroups of G such that the inclusion $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence. There is an isomorphism

$$T(G, S) \cong \lim_{[P_0 < \ldots < P_n]} \mathrm{H}^1(N_G(P_0 < \ldots < P_n); k^\times),$$

  where the limit is indexed over chains of subgroups in $\mathscr{C}$, ordered by refinement. [Gro18, Theorem D]
- Let $\mathscr{C}$ be as above and let $\mathrm{H}^*_G(X; F)$ denote the G-equivariant Bredon cohomology of a G-space X with coefficients in the functor F (see Section 5). There is an isomorphism $T(G, S) \cong \mathrm{H}^0_G(\mathscr{C}; \mathrm{H}^1(-; k^\times))$. [Gro18, §5]

What happens in the exceptional cases, when $T(S)$ has torsion, is also completely understood: Carlson, Mazza, and Thévenaz [MT07; CMT13] show that the restriction map $T(G) \to T(S)$ is a surjection; in particular, all endotrivial S-modules lift to $T(G)$ and hence are G-stable. Furthermore, apart from the (well-understood) cyclic case, the restriction map is split, so $TT(G) \cong T(G,S) \oplus TT(S)$. This allows for a full description of $TT(G)$ from that of $T(G,S)$, using the classification of $T(S)$.

2.10. **Picard spaces.** The group $T(G)$ is in fact the *Picard group* of StMod_{kG}, *i.e.* the group of equivalence classes of $\otimes$-invertible objects. The Picard group construction provides a functor from symmetric monoidal categories to abelian groups.

One unfortunate property of the Picard group is that it does not preserve (homotopy) limits of categories, so is hard to calculate using descent methods. This can be remedied by considering instead the *Picard space* $\mathrm{Pic}(\mathcal{C})$ of a symmetric monoidal ∞-category $\mathcal{C}$: this is the ∞-groupoid (*i.e.* space) of $\otimes$-invertible objects in $\mathcal{C}$ and equivalences between them. Its set of connected components can be identified with the Picard group of $\mathcal{C}$, but due to its extra structure, Pic commutes with homotopy limits as a functor to pointed spaces. More information can be found in [MS16, §2].

For any symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$, we have a description of the higher homotopy groups of $\mathrm{Pic}(\mathcal{C})$. These are given by

$$\pi_i \operatorname{Pic}(\mathcal{C}) \cong \begin{cases} \operatorname{PicGp}(\mathcal{C}) & \text{when } i = 0, \\ (\pi_0 \underline{\operatorname{Hom}}(\mathbb{1}, \mathbb{1}))^\times & \text{when } i = 1, \text{ and} \\ \pi_{i-1} \underline{\operatorname{Hom}}(\mathbb{1}, \mathbb{1}) & \text{when } i \geq 2, \end{cases}$$

where PicGp denotes the Picard group. In the case of StMod_{kG}, we have

$$\pi_i \operatorname{Pic}(\operatorname{StMod}_{kG}) \cong \begin{cases} T(G) & \text{when } i = 0, \\ k^\times & \text{when } i = 1, \text{ and} \\ \hat{\mathrm{H}}^{1-i}(G;\, k) & \text{when } i \geq 2, \end{cases} \tag{2.11}$$

where $\hat{\mathrm{H}}^n(G;\, M) \cong \pi_0(\underline{\operatorname{Hom}}_G(\Omega^n k, M))$ denotes the Tate cohomology of G.

3. Decomposition of the stable module ∞-category over the orbit category

We continue to let G be a finite group. A key input for deriving our obstruction will be a symmetric monoidal decomposition of the stable module ∞-category as a homotopy limit over the orbit category. Let $\mathrm{Cat}_\infty^\otimes$ be the ∞-category of symmetric monoidal ∞-categories. There are various ways to construct the required functor $\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$; one possibility is to follow the approach of [Mat16, §9.5] and use the composite of the functors

$$\begin{array}{rcl} \mathcal{O}(G)^{\mathrm{op}} & \to & \operatorname{CAlg}(\operatorname{StMod}_{kG}) \\ G/H & \mapsto & \prod_{G/H} k \end{array}$$

and

$$\begin{array}{rrcl} \operatorname{Mod}(-)\colon & \operatorname{CAlg}(\operatorname{StMod}_{kG}) & \to & \mathrm{Cat}_\infty^\otimes \\ & A & \mapsto & \operatorname{Mod}_{\operatorname{StMod}_{kG}}(A) \end{array}$$

along with Theorem 2.5. Here $\mathrm{CAlg}(\mathcal{C})$ denotes the ∞-category of commutative algebra objects in a symmetric monoidal ∞-category $\mathcal{C}$. Note that, by [Lur17, Proposition 3.2.2.1], the ∞-category $\mathrm{CAlg}(\mathcal{C})$ has all homotopy limits which exist in $\mathcal{C}$ and these are preserved by the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$.

3.1. **Proposition.** *Let $\mathscr{C}^e$ be a collection of subgroups of G that is closed under intersection and such that every elementary abelian p-subgroup of G is contained in a subgroup in $\mathscr{C}^e$. Let $\mathscr{C}$ denote the non-trivial subgroups in $\mathscr{C}$. There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \operatorname*{holim}_{G/H\in\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kH},$$

where the homotopy limit is taken in the ∞-category of symmetric monoidal ∞-categories.

Proof. Corollary 9.16 in [Mat16] implies that

$$\mathrm{StMod}_{kG} \simeq \operatorname*{holim}_{\mathcal{O}_{\mathscr{C}^e}(G)^{\mathrm{op}}} \mathrm{StMod}_{kH}.$$

We note that $\mathrm{StMod}_{k\{1\}}$ is trivial, so we can remove $G/\{1\}$ from the diagram without changing the value of the homotopy limit: indeed, if we let $F : \mathcal{O}_{\mathscr{C}^e}(G)^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ denote the functor constructed above and $i : \mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{O}_{\mathscr{C}^e}(G)^{\mathrm{op}}$ denote the inclusion, then F is equivalent to the homotopy right Kan extension along i of the restriction i^*F. The homotopy limit of $\mathrm{Ran}_i\, i^*F$ agrees with the homotopy limit of i^*F, because Ran_i is right adjoint to i^* and taking homotopy limits is right adjoint to the constant diagram functor. Therefore, we get

$$\mathrm{StMod}_{kG} \simeq \operatorname*{holim}_{\mathcal{O}_{\mathscr{C}^e}(G)^{\mathrm{op}}} \mathrm{Ran}_i\, i^*F \simeq \operatorname*{holim}_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} i^*F. \qquad \square$$

The Picard space functor commutes with homotopy limits [MS16, Proposition 2.2.3], so we get a decomposition of pointed simplicial sets

$$\mathrm{Pic}(\mathrm{StMod}_{kG}) \simeq \operatorname*{holim}_{G/H\in\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{Pic}(\mathrm{StMod}_{kH}).$$

The homomorphism $T(G) \to \lim_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} T(-)$ is exactly the induced map

$$\pi_0\,\mathrm{Pic}(\mathrm{StMod}_{kG}) \to \lim_{G/H\in\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \pi_0\,\mathrm{Pic}(\mathrm{StMod}_{kH}).$$

4. Obstruction theory for homotopy limits

In [Bou89, §5.2], Bousfield gives an obstruction theory for cosimplicial spaces that we can apply to the problem of lifting from $\lim_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \pi_0\,\mathrm{Pic}(\mathrm{StMod}_{kP})$ to $\pi_0\,\mathrm{holim}_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{Pic}(\mathrm{StMod}_{kP})$, *i.e.* from $\lim_{\mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} T(-)$ to $T(G)$. We recall this obstruction theory in the case of a diagram of Kan complexes with trivial action of their fundamental group on their higher homotopy groups, then specialise to the case of interest.

Let sSet be the category of simplicial sets and let csSet be the category of cosimplicial spaces. Given a diagram of Kan complexes $F : I^{\mathrm{op}} \to \mathrm{sSet}$, let $X^\bullet$ denote the the fibrant cosimplicial space given by

$$X^n := \prod_{i_0\to\ldots\to i_n} F(i_0), \tag{4.1}$$

where the product is indexed by the n-simplices in the nerve of I. The space $\mathrm{Tot}\, X^\bullet$ is a model for the homotopy limit of F, where $\mathrm{Tot}\, X^\bullet$ denotes the simplicial set $\mathrm{Map}_{\mathrm{csSet}}(\Delta^\bullet, X^\bullet)$. This model is itself a limit of the tower of fibrations

$$\ldots \to \mathrm{Tot}_n\, X^\bullet \to \mathrm{Tot}_{n-1}\, X^\bullet \to \ldots \to \mathrm{Tot}_0\, X^\bullet,$$

where $\mathrm{Tot}_n\, X^\bullet$ denotes the simplicial set $\mathrm{Map}_{\mathrm{csSet}}(\mathrm{sk}_n\,\Delta^\bullet, X^\bullet)$.

Assume that $\pi_1(X^s, x)$ acts trivially on $\pi_n(X^s, x)$ for all $n \geq 1$, $s \geq 0$ and $x \in X^s$. If we choose a vertex $v_r \in \mathrm{Tot}_r\, X^\bullet$ that lifts to $\mathrm{Tot}_{r+1}\, X^\bullet$, then Bousfield's obstruction theory gives an obstruction class in $\pi^{r+2}\pi_{r+1}X^\bullet$ that vanishes if and only if v_r is liftable to $\operatorname{Tot}_{r+2} X^\bullet$ (this is explained in more detail in the proof of the following lemma). Here $\pi^r\pi_t X^\bullet$ denotes the r^{th} *cohomotopy group* of $\pi_t X^\bullet$, defined to be the r^{th} cohomology group of the cochain complex given by the *normalised homotopy groups*

$$N\pi_t(X^\bullet, v) := \bigcap_{i=0}^{\bullet-1} \ker(s^j : \pi_t(X^\bullet, v) \to \pi_t(X^{\bullet-1}, s^j v))$$

and the differential induced by the alternating sum of the face maps.

4.2. **Lemma.** *Let $\mathscr{C}$ be a collection for which we have a decomposition*

$$\operatorname{StMod}_{kG} \xrightarrow{\sim} \operatorname*{holim}_{\mathcal{O}_{\mathscr{C}}(G)^{\text{op}}} \operatorname{StMod}_{kH}$$

and let $(M_H)_{G/H\in\mathcal{O}_{\mathscr{C}}(G)} \in \lim_{\mathcal{O}_{\mathscr{C}}(G)^{\text{op}}} \pi_0 \operatorname{Pic}(\operatorname{StMod}_{k-})$. There are obstructions to lifting (M_H) to an element of $\pi_0 \operatorname{Pic}(\operatorname{StMod}_{kG})$ that lie in $\mathrm{H}^{r+2}(\mathcal{O}_{\mathscr{C}}(G); \pi_{r+1} \operatorname{Pic}\operatorname{StMod}_{k-})$ for $r \geq 0$. When $r \geq 1$, these obstruction groups identify with $\mathrm{H}^{r+2}(\mathcal{O}_{\mathscr{C}}(G); \hat{\mathrm{H}}^{-r}(-; k))$.

Proof. The Picard space is by construction a Kan complex, and the actions of its fundamental group on its higher homotopy groups are trivial since it has an E_∞-space structure. We apply Bousfield's obstruction theory: by hypothesis, we start with a vertex $v_0 \in \operatorname{Tot}_0 X^\bullet$ that lifts to $\operatorname{Tot}_1 X^\bullet$, where $X^\bullet$ is the cosimplicial space defined in (4.1) associated with the functor

$$\operatorname{Pic}(\operatorname{StMod}_{k-}) : \mathcal{O}_{\mathscr{C}}(G)^{\text{op}} \to \mathrm{sSet},$$

whose totalisation computes the homotopy limit $\operatorname{holim}_{\mathcal{O}_{\mathscr{C}}(G)^{\text{op}}} \operatorname{Pic}(\operatorname{StMod}_{k-})$. The trivial action of the fundamental group implies that this case is covered by the spectral sequence of [Bou89, §2.6], and [Bou89, §5.2] gives an obstruction class in $\pi^2\pi_1 X^\bullet$ that vanishes if and only if v_0 lifts further to $\operatorname{Tot}_2 X^\bullet$. Now let $r \geq 1$ and $v_r \in \operatorname{Tot}_r X^\bullet$ be a vertex liftable to $\operatorname{Tot}_{r+1} X^\bullet$. [Bou89, §5.2] gives an obstruction in $\pi^{r+2}\pi_{r+1}X^\bullet$ that vanishes if and only if v_r lifts further to $\operatorname{Tot}_{r+2} X^\bullet$. (In particular, we do not require any hypotheses on the vanishing of Whitehead products in this case.)

The obstruction group $\pi^{r+2}\pi_{r+1}X^\bullet$ is isomorphic to $\mathrm{H}^{r+2}(\mathcal{O}_{\mathscr{C}}(G); \pi_{r+1} \operatorname{Pic}\operatorname{StMod}_{k-})$ by [GJ09, VIII.2 (2.18)], so it remains for us to check that the homotopy groups of $\operatorname{Pic}(\operatorname{StMod}_{kH})$ are as we claimed earlier in Equation (2.11). Since $\Omega \operatorname{Pic}(\operatorname{StMod}_{kH}) \simeq \operatorname{Aut}(k)$, we have

$$\pi_{r+1} \operatorname{Pic}(\operatorname{StMod}_{kH}) \cong \pi_r \operatorname{Aut}(k),$$

and hence $\pi_1 \operatorname{Pic}(\operatorname{StMod}_{kH}) \cong k^\times$. For $r \geq 1$ we have

$$\pi_{r+1} \operatorname{Pic}(\operatorname{StMod}_{kH}) \cong \pi_r \underline{\operatorname{Hom}}(k, k) \cong \pi_0(\underline{\operatorname{Hom}}(\Omega^{-r}k, k)) \cong \hat{\mathrm{H}}^{-r}(H; k). \qquad \square$$

Since k is a field of characteristic p, a result of Jackowski and McClure shows that all but the first of these obstruction groups are zero when $\mathscr{C} = \mathcal{S}_p(G)$:

4.3. **Lemma.** *Let $j \in \mathbb{Z}$. The functor $\hat{\mathrm{H}}^j(-; k) : \mathcal{O}_{\mathcal{S}}(G)^{\text{op}} \to \operatorname{Mod}_k$ is acyclic,* i.e.

$$\mathrm{H}^i(\mathcal{O}_{\mathcal{S}}(G); \hat{\mathrm{H}}^j(-; k)) = 0$$

for any $i \geq 1$.

Proof. We again let $\mathcal{O}_p(G)$ denote the orbit category on all p-subgroups of G, and consider the extended functor $F := \hat{\mathrm{H}}^j(-; k) : \mathcal{O}_p(G)^{\text{op}} \to \operatorname{Mod}_k$. Proposition 5.14 in [JM92] says that any proto-Mackey functor $\mathcal{O}_p(G)^{\mathrm{op}} \to \mathrm{Mod}_{\mathbb{Z}_{(p)}}$ is acyclic, and we observe that F is indeed a proto-Mackey functor. This is well-known, with the covariant part of the Mackey functor sending a morphism $g\colon G/P \to G/Q$ to the transfer map

$$\begin{aligned}\hat{\mathrm{H}}^j(P;\, k) \cong \pi_0\, \underline{\mathrm{Hom}}_P(\mathrm{res}_g\, \Omega^j k, \mathrm{res}_g\, k) &\cong \pi_0\, \underline{\mathrm{Hom}}_Q(\Omega^j k, \mathrm{ind}_g\, \mathrm{res}_g\, k) \\ &\xrightarrow{\varepsilon_*} \pi_0\, \underline{\mathrm{Hom}}_Q(\Omega^j k, k) \cong \hat{\mathrm{H}}^j(Q;\, k).\end{aligned}$$

The Mackey decomposition formula for F follows from the Mackey decomposition theorem [Ben98, Theorem 3.3.4] applied to k.

Next we need to deduce acyclicity of $F' := \hat{\mathrm{H}}^j(-;\, k)\colon \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathrm{Mod}_k$ from acyclicity of F: that is, we need to remove the trivial subgroup again. Since right Kan extension along $\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathcal{O}_p(G)^{\mathrm{op}}$ extends by the zero group on the coset G/e, it follows that F is the right Kan extension of F'. It also follows that taking right Kan extensions preserves epimorphisms. By [JM92, Lemma 3.1], we have $\mathrm{H}^i(\mathcal{O}_{\mathcal{S}}(G);\, F') \cong \mathrm{H}^i(\mathcal{O}_p(G);\, F)$ for all i, so F' is also acyclic. ☐

4.4. *Remark.* Note that the functors considered in the vanishing result [JM92, Proposition 5.14] are implicitly contravariant. It is also important that we consider the orbit category on all p-subgroups $\mathcal{O}_{\mathcal{S}}(G)$ rather than, for example, restricting to the orbit category on elementary abelian p-subgroups $\mathcal{O}_{\mathcal{A}}(G)$, where in general the higher limits do not vanish; see [Gro02, Remark 3.4]. (As in the above proof, the inclusion or exclusion of the trivial subgroup is irrelevant here.)

We can illustrate the latter remark by the example $G = D_8$, making forward reference to some results later in the paper. There are two non-conjugate elementary abelian subgroups of G, which we will call E_1 and E_2. The limit $\lim_{\mathcal{O}_{\mathcal{A}}(G)^{\mathrm{op}}} T(-)$ then identifies with $T(E_1) \oplus T(E_2) \cong \mathbb{Z}^2$. By Corollary 7.12, the obstruction appearing in Theorem 4.5 below vanishes when $p = 2$. Since G is itself a 2-group, $T(G, S) = 0$. Therefore, if all the higher obstructions vanished, we would have an isomorphism $T(G) \xrightarrow{\sim} T(E_1) \oplus T(E_2)$ induced by restriction. Since the restriction from $T(G) \to T(E_1) \oplus T(E_2) \cong \mathbb{Z}^2$ lands only in the "even" part of the lattice [CT00, Theorem 5.4], one of the groups $\mathrm{H}^{r+2}(\mathcal{O}_{\mathcal{A}}(G);\, \hat{\mathrm{H}}^{-r}(-;\, k))$ must be non-zero. We can nevertheless hope to extract useful information from the decomposition over $\mathcal{O}_{\mathcal{A}}(G)$; see Remark 4.8 for an example.

4.5. **Theorem.** *There is an exact sequence*

$$0 \to T(G, S) \to T(G) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \xrightarrow{\alpha} \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^\times),$$

and we can explicitly describe a 2-cocycle that represents α: given an element $(M_P)_{G/P \in \mathcal{O}_{\mathcal{S}}(G)} \in \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$, choose an equivalence $\lambda_g\colon \mathrm{res}_g\, M_Q \to M_P$ in StMod_{kP} *for every morphism $g\colon G/P \to G/Q$ in $\mathcal{O}_{\mathcal{S}}(G)$. We obtain a potentially non-commutative diagram of equivalences*

$$\begin{array}{ccccc} & & \mathrm{res}_g\, \mathrm{res}_h\, M_R & & \\ & \swarrow^{\mathrm{res}_g \lambda_h} & & \searrow^{\lambda_{gh}} & \\ \mathrm{res}_g\, M_Q & & \xrightarrow{\quad \lambda_g \quad} & & M_P \end{array}$$

for every $G/P \xrightarrow{g} G/Q \xrightarrow{h} G/R$ in $\mathcal{O}_{\mathcal{S}}(G)$. We define $\alpha_M(g, h)$ to be the element of $k^\times$ that corresponds to the automorphism

$$\mathrm{res}_g\, \mathrm{res}_h\, M_R \xrightarrow{\lambda_{gh}} M_P \xleftarrow{\lambda_g} \mathrm{res}_g\, M_Q \xleftarrow{\mathrm{res}_g \lambda_h} \mathrm{res}_g\, \mathrm{res}_h\, M_R$$

determined by the above diagram. (Recall from Lemma 2.7 that $\pi_0 \underline{\mathrm{Aut}}(M) \cong k^\times$ for any endotrivial module M.) The cohomology class $\alpha\big((M_P)\big)$ is represented by α_M.

4.6. **Notation.** When the G-stable endotrivial S-module (M_P) is clear from context, we will drop the subscript on α_M and use α to refer to both the homomorphism and the cocycle.

Proof. Lemma 4.3 shows that all the obstructions given in Lemma 4.2 vanish with the possible exception of the obstruction in $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$. This establishes the existence of the exact sequence, so we just need to show that α_M actually represents α.

Let $X^\bullet$ be the cosimplicial space defined in (4.1) associated with the functor

$$\mathrm{Pic}(\mathrm{StMod}_{k-}) : \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathrm{sSet}.$$

Section 5.1 in [Bou89] describes the obstruction in $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$: if we have chosen a vertex $b \in \mathrm{Tot}_1 X^\bullet$ then we obtain a map $c(b) : \partial\Delta^2 = \mathrm{sk}_1 \Delta^2 \to X^2$ in the normalised homotopy $N\pi_1(X^2, b)$ that represents the obstruction in $\pi^2\pi_1(X^\bullet, b)$.

Choosing a vertex in $\mathrm{Tot}_1 X^\bullet$ is the same as choosing endotrivial modules $M_P \in \mathrm{StMod}_{kP}$ for all $G/P \in \mathcal{O}_{\mathcal{S}}(G)$ along with equivalences $\lambda_g : \mathrm{res}_g M_Q \to M_P$ for all $g : G/P \to G/Q$. The obstruction $c(b)$ corresponding to this choice of vertex is then the 2-cocycle described in the statement of the theorem. □

4.7. *Remark.* In Appendix A, we compare the obstruction α constructed above with an obstruction that appeared in work of Balmer [Bal15].

4.8. *Remark.* We outline how Proposition 3.1 should allow us to re-derive Alperin's formula for the torsion-free rank of $T(G)$. Let n_G denote the number of conjugacy classes of rank 2 maximal elementary abelian subgroups of G. Recall that the torsion-free rank of $T(G)$ is given by

$$\dim_{\mathbb{Q}}(T(G) \otimes \mathbb{Q}) = \begin{cases} 0 & \text{if the } p\text{-rank of } G \text{ is one,} \\ n_G & \text{if the } p\text{-rank of } G \text{ is two, and} \\ n_G + 1 & \text{if the } p\text{-rank of } G \text{ is three or greater.} \end{cases}$$

This was proved for finite groups by Carlson–Mazza–Nakano [CMN06, Theorem 3.1] and for p-groups by Alperin [Alp01, Theorem 4].

We use Proposition 3.1 to decompose StMod_{kG} over $\mathcal{O}_{\mathcal{A}}(G)$:

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \underset{\mathcal{O}_{\mathcal{A}}(G)^{\mathrm{op}}}{\mathrm{holim}} \mathrm{StMod}_{kV}.$$

Since $\mathcal{O}_{\mathcal{A}}(G)$ is a finite 1-category, Alperin's formula holds for $\lim_{\mathcal{O}_{\mathcal{A}}(G)^{\mathrm{op}}} T(-)$: rationalisation commutes with finite limits, and the group-theoretic argument given prior to Lemma 4 of [Alp01] shows that any two elementary abelian subgroups of p-rank at least three are connected by a zig-zag of elementary abelian subgroups of p-rank at least two. Therefore, $\lim_{\mathcal{O}_{\mathcal{A}}(G)^{\mathrm{op}}}(T(-) \otimes \mathbb{Q})$ has one copy of $\mathbb{Q}$ for each conjugacy class of rank two maximal elementary abelian subgroups, plus an extra copy if there are any elementary abelian subgroups of p-rank at least three. Lemma 4.2 implies that $T(G)$ is obtained from $\lim_{\mathcal{O}_{\mathcal{A}}(G)^{\mathrm{op}}} T(-)$ by repeatedly taking the kernel of a homomorphism to $\mathrm{H}^{r+2}(\mathcal{O}_{\mathcal{A}}(G); \pi_{r+1} \mathrm{Pic}\,\mathrm{StMod}_{k-})$, so it is enough to show that this procedure preserves the torsion-free rank.

We now come to a gap in the argument: it is not clear that the obstructions given in [Bou89, §5.2] arise as the differentials in a spectral sequence of the form constructed in [Bou89, §§2.4–2.6]. Assuming that this is nevertheless the case, the analogue for the stable module ∞-category of [MNN19, Theorem 2.25] shows that this spectral sequence has a horizontal vanishing line at a finite stage. Note that the nilpotency condition of that theorem is satisfied for any module in StMod_{kG} because the algebra object $\prod_{V \in \mathcal{A}_p(G)} A_V^G$ is descendable. (Recall that A_V^G denotes

$\mathrm{coind}_V^G(k)$; see Theorem 2.5.) It seems likely that the domain of definition of the spectral sequences can be expanded as claimed, at least if one is willing to construct a less general spectral sequence than in [Bou89].

Therefore, there are only finitely many obstruction groups to consider, and it is enough to check that each of them is torsion. For the obstruction in $\mathrm{H}^2(\mathcal{O}_{\mathcal{A}}(G); k^\times)$, we have rational equivalences

$$\mathrm{H}_*(\mathcal{O}_{\mathcal{A}}(G)) \otimes \mathbb{Q} \xleftarrow{\cong} \mathrm{H}_*(\mathcal{T}_{\mathcal{A}}(G)) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{H}_*(\mathcal{A}_p(G)/G) \otimes \mathbb{Q}$$

by [Gro18, Proposition 4.33]. Symonds' theorem [Sym98] says that $\mathcal{S}_p(G)/G$ is contractible, so $\mathcal{A}_p(G)/G$ is as well. Since $\mathrm{H}_i(\mathcal{O}_{\mathcal{A}}(G))$ is finitely generated, we deduce that it is finite when $i > 0$. The universal coefficient theorem then implies that $\mathrm{H}^2(\mathcal{O}_{\mathcal{A}}(G); k^\times)$ is also finite provided k is algebraically closed. The argument for the higher obstruction groups is simpler: each $\mathrm{H}^{r+2}(\mathcal{O}_{\mathcal{A}}; \hat{\mathrm{H}}^{-r}(-; k))$ is a k-vector space and hence is annihilated by multiplication by p.

5. An obstruction in Bredon cohomology

While useful conceptually, the obstruction α of Theorem 4.5 can be difficult to work with in practice. The category $\mathcal{O}_{\mathcal{S}}(G)$ contains the Weyl group of every p-subgroup of G, so it can be hard to compute its cohomology for specific choices of G that we are interested in. In this section, we show that the obstruction α in $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$ vanishes if and only if another obstruction β in the Bredon cohomology group $\mathrm{H}^1_G(\mathcal{S}_p(G); \mathrm{H}^1(-; k^\times))$ vanishes (Proposition 5.9).

This latter obstruction has several advantages for practical computations: for example, $\mathcal{S}_p(G)$ is a finite poset, so computing the obstruction group is much easier. We can also replace $\mathcal{S}_p(G)$ with a smaller G-homotopy equivalent poset, such as the poset $\mathcal{B}_p(G)$ of non-trivial p-radical subgroups or the poset $\mathcal{A}_p(G)$ of non-trivial elementary abelian p-subgroups. It will later allow us to provide an alternative viewpoint on the question of lifting: given a G-stable endotrivial S-module $(M_P)_{G/P \in \mathcal{O}_{\mathcal{S}}(G)}$, there is a lift of M_S to $T(G)$ if and only if we can specify one-dimensional characters, one for every elementary abelian p-subgroup $P \in \mathcal{A}_p(G)$, such that they satisfy a certain compatibility condition; see Section 7.

We start by recalling the isotropy spectral sequence as defined in [Bro82, VII (5.3)] or [Dwy98, §2.3]. Given a G-space X and an abelian group A, we have a quasi-isomorphism

$$\mathrm{C}^\bullet(X_{hG}; A) \simeq \mathrm{Tot}^\bullet(\mathrm{Hom}_G(\mathrm{C}_\bullet(\mathrm{E}G), \mathrm{C}^\bullet(X; A))). \tag{5.1}$$

Here $\mathrm{C}^\bullet(X; A)$ denotes the cochains on X with coefficients in A; similarly, $\mathrm{C}_\bullet(X; A)$ denotes the chains on X with coefficients in A. We can filter the right-hand side of Equation (5.1) by skeleta of X, giving rise to a spectral sequence calculating the cohomology $\mathrm{H}^*(X_{hG}; A)$. More explicitly, we use the filtration

$$\ldots \subseteq F_1^\bullet \subseteq F_0^\bullet = \mathrm{C}^\bullet(X_{hG}; A) \quad \text{with} \quad F_r^n := \bigoplus_{\substack{s+t=n \\ s \geq r}} \mathrm{Hom}_G(\mathrm{C}_t(\mathrm{E}G), \mathrm{C}^s(X; A)).$$

The corresponding filtered complex spectral sequence is isomorphic to the double complex spectral sequence where one first takes the differential in the $C_\bullet(\mathrm{E}G)$ direction. Therefore, the spectral sequence has $E_1^{\bullet\bullet}$ page given by

$$E_1^{s,t} = \mathrm{H}^t(G; \mathrm{C}^s(X; A))$$

with differential induced by the differential in $\mathrm{C}^\bullet(X; A)$.

The rows of the $E_1^{\bullet\bullet}$ page identify with the cochain complexes $\mathrm{C}^s_G(X;\, \mathrm{H}^t(-;\, A))$, where $\mathrm{C}^\bullet_G(X;\, F)$ denotes the *G-equivariant Bredon cochains* of X with coefficients in some functor $F\colon\, \mathcal{O}(G)^{\mathrm{op}} \to \mathrm{Ab}$. This is defined by

$$\mathrm{C}^s_G(X;\, F) := \Big(\prod_{\sigma\in X_s} F(G_\sigma)\Big)^G, \tag{5.2}$$

where G_σ is the stabiliser of σ in G. The *G-equivariant Bredon cohomology* of X with coefficients in F, denoted by $\mathrm{H}^*_G(X;\, F)$, is then defined to be the cohomology of the above cochain complex, which is an invariant of the G-equivariant homotopy type of X; see [Bre67, §I.6]. In this way, we obtain the *isotropy spectral sequence*

$$E_2^{s,t} \cong \mathrm{H}^s_G(X;\, \mathrm{H}^t(-;\, A)) \Rightarrow \mathrm{H}^{s+t}(X_{hG};\, A). \tag{5.3}$$

Let $\mathscr{C}$ be a collection. Thomason's theorem [Tho79, Theorem 1.2] implies that $\mathcal{T}_{\mathscr{C}}$ is homotopy equivalent to $\mathscr{C}_{hG}$. Therefore, when applied to $\mathscr{C}$, the isotropy spectral sequence computes the cohomology of $\mathcal{T}_{\mathscr{C}}$ in terms of the Bredon cohomology of $\mathscr{C}$.

The canonical quotient functor $\mathcal{T}_{\mathcal{S}}(G) \to \mathcal{O}_{\mathcal{S}}(G)$ induces an injective map

$$\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^\times) \to \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times) \tag{5.4}$$

that identifies $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^\times)$ with a direct summand of $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$, by [Gro18, Corollary 4.36]. For certain collections $\mathscr{C}$, we can also identify $\mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^\times))$ with a subgroup of $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$:

5.5. **Lemma.** *There is a natural injective map*

$$\mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^\times)) \to \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$$

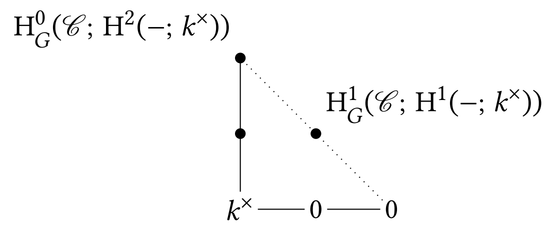
for any collection $\mathscr{C}$ of non-trivial p-subgroups such that $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence.

Proof. Consider the isotropy spectral sequence (5.3) for $X = \mathscr{C}$. Symonds' theorem [Sym98] says that $\mathcal{S}_p(G)/G$ is contractible, and hence so is $\mathscr{C}/G$. Therefore for $s > 0$ we have

$$E_2^{s,0} \cong \mathrm{H}^s_G(\mathscr{C};\, \mathrm{H}^0(-;\, k^\times)) \cong \mathrm{H}^s(\mathscr{C}/G;\, k^\times) = 0,$$

and the bottom-left corner of the E_2 page of the spectral sequence is as follows:

(5.6)



We deduce that $E_2^{1,1} \cong \mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^\times))$ injects into $\mathrm{H}^2(\mathcal{T}_{\mathscr{C}}(G);\, k^\times)$. Since $\mathscr{C}$ is G-homotopy equivalent to $\mathcal{S}_p(G)$, Thomason's theorem implies that $\mathcal{T}_{\mathscr{C}}(G)$ is homotopy equivalent to $\mathcal{T}_{\mathcal{S}}(G)$. □

5.7. *Remark.* We can describe the image γ_M of α_M in $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$ analogously to the description of α_M in Theorem 4.5: for every $g^{-1}\colon\, P \to Q$ in $\mathcal{T}_{\mathcal{S}}(G)$, we choose an equivalence $\lambda_g\colon\, g \otimes_Q M_Q \to M_P$ in StMod_{kP}, and then for every 2-chain

$$P \xrightarrow{g^{-1}} Q \xrightarrow{h^{-1}} R$$

we define $\gamma_M(g^{-1}, h^{-1})$ to be the element of $k^\times$ that corresponds to the automorphism

$$gh \otimes_R M_R \xrightarrow{g\otimes\lambda_h} g \otimes_Q M_Q \xrightarrow{\lambda_g} M_P \xleftarrow{\lambda_{gh}} gh \otimes_R M_R$$

of $gh\otimes_R M_R$. That γ_M represents the same cohomology class as the image of α_M in $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A_Q^G) & \xrightarrow{\mathrm{res}_g} & \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A_P^G) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{StMod}_{kQ} & \xrightarrow{g\otimes_P -} & \mathrm{StMod}_{kP} \end{array}$$

for every $g^{-1} : P \to Q$ in $\mathcal{T}_{\mathcal{S}}(G)$. This can be checked using the description of the vertical maps given in Construction 5.23 of [MNN17]. Here A_P^G denotes the algebra object $\mathrm{coind}_P^G k$ (recall Theorem 2.5). Intuitively, there is a canonical choice of equivalence $x \otimes_P M_P \to M_P$ for every $x \in P$, given by the action of x on M_P, so the difference between the orbit category and the transport category is not relevant for our obstruction class.

The injectivity of the homomorphism $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^\times) \to \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times)$ combined with Theorem 4.5 implies that we also have an exact sequence

$$0 \to T(G,S) \to T(G) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \xrightarrow{\gamma} \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times).$$

We use this to show that we can always lift in the following special case:

5.8. **Proposition.** *Suppose that G has a non-trivial normal p-subgroup. Any G-stable endotrivial S-module lifts to a module in $T(G)$.*

Proof. Let P be a non-trivial normal p-subgroup of G. We consider the following commutative diagram:

$$\begin{array}{ccccc} T(G) & \longrightarrow & \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) & \xrightarrow{\gamma} & \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^\times) \\ & \searrow & \downarrow & & \downarrow \cong \\ & & T(Z(P)) & \xrightarrow{\gamma'} & \mathrm{H}^2(G;\, k^\times) \end{array}$$

The right-hand map is restriction to the full subcategory on the object $Z(P)$. It is an isomorphism because $\mathcal{T}_{\mathcal{S}}(G)$ is homotopy equivalent to $\mathcal{S}_p(G)_{hG}$ and $\mathcal{S}_p(G)$ is G-equivariantly contractible to the object $Z(P)$. The map γ' is defined similarly to γ, but again restricted to automorphisms of the object $Z(P)$. Since neither γ nor γ' depend on the choices of equivalences that appear in their respective constructions, the right-hand square commutes.

By Dade's Theorem 2.8, every module in $T(Z(P))$ is equivalent to $\Omega^n k$ for some n, and hence the diagonal map from $T(G)$ is surjective. Since the top composite is zero, this implies that γ' is also zero. The proposition now follows from exactness of the top row. □

5.9. **Proposition.** *Let $\mathscr{C}$ be any collection of non-trivial p-subgroups such that $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence. There is a homomorphism β making the diagram*

$$\begin{array}{ccc} \lim\limits_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) & \xrightarrow{\ \alpha\ } & \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^{\times}) \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow \\ \mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^{\times})) & \hookrightarrow & \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^{\times}) \end{array}$$

commute.

Proof. Recall the form of the E_2-page of the isotropy spectral sequence for $\mathscr{C}$, as shown in Diagram 5.6. To show that a class $\theta \in \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^{\times})$ lies in the subgroup $\mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^{\times}))$, it is enough to show that it maps to zero in $\mathrm{H}^0_G(\mathscr{C};\, \mathrm{H}^2(-;\, k^{\times}))$. Corollary 7.2 in [Gro02] shows that $\mathrm{H}^0_G(\mathscr{C};\, \mathrm{H}^2(-;\, k^{\times}))$ identifies with $\lim_{\bar{s}S_{\mathscr{C}}(G)} \mathrm{H}^2(N_G(-);\, k^{\times})$, where $\bar{s}S_{\mathscr{C}}(G)$ denotes the poset of G-conjugacy classes of chains of subgroups of $\mathscr{C}$ with the order relation given by refinement, so it is enough to check that θ maps to the zero class in $\mathrm{H}^2(N_G(\sigma);\, k^{\times})$ for every $\sigma \in \bar{s}S_{\mathscr{C}}(G)$. If $\sigma = [P_0 < \ldots < P_n]$ then the image of θ in $\mathrm{H}^2(N_G(\sigma);\, k^{\times})$ is given by restriction to $N_G(\sigma) \leq \mathrm{Aut}_{\mathcal{T}_{\mathcal{S}}(G)}(P_0)$.

For the class γ_M that denotes the image of α_M in $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^{\times})$, we can immediately reduce to the case where $G = N_G(\sigma)$ by considering the diagram

$$\begin{array}{ccc} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) & \xrightarrow{\ \gamma\ } & \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^{\times}) \\ \Big\downarrow & & \Big\downarrow \\ \lim_{\mathcal{O}_{\mathcal{S}}(N_G(\sigma))^{\mathrm{op}}} T(-) & \xrightarrow{\ \gamma\ } & \mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(N_G(\sigma));\, k^{\times}) \end{array}$$

and observing that the restriction $\mathrm{H}^2(\mathcal{T}_{\mathcal{S}}(G);\, k^{\times}) \to \mathrm{H}^2(N_G(\sigma);\, k^{\times})$ factors through the right-hand map of this diagram. By Proposition 5.8, the bottom map of the square is zero, and hence so is the restriction of γ_M to $\mathrm{H}^2(N_G(\sigma);\, k^{\times})$.

We have now shown that the obstruction class $\alpha_M \in \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^{\times})$ maps to zero in the zeroth Bredon cohomology group $\mathrm{H}^0_G(\mathscr{C};\, \mathrm{H}^2(-;\, k^{\times}))$. Therefore, there is a homomorphism β as claimed. □

5.10. **Corollary.** *There is an exact sequence*

$$0 \to T(G,S) \to T(G) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \xrightarrow{\beta} \mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^{\times}))$$

for any collection $\mathscr{C}$ of non-trivial p-subgroups such that $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence.

We can give an explicit description of the obstruction in Bredon cohomology:

5.11. **Proposition.** *Let $\mathscr{C}$ be a collection of non-trivial p-subgroups such that $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence. The homomorphism*

$$\beta : \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \to \mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^{\times}))$$

has the following explicit description. Let $(M_P)_{G/P \in \mathcal{O}_{\mathscr{C}}(G)}$ be a G-stable endotrivial S-module. Choose equivalences $\lambda^P_g :\ g \otimes_P M_P \to M_P$ in StMod_{kP} for every $P \in \mathscr{C}$ and $g \in N_G(P)$, such that

$$(5.12)\qquad \begin{array}{ccc} gh \otimes_P M_P & \xrightarrow{\ g \otimes_P \lambda^P_h\ } & g \otimes_P M_P \\ & {\scriptstyle \lambda^P_{gh}} \searrow \quad \swarrow {\scriptstyle \lambda^P_g} & \\ & M_P & \end{array}$$

commutes in the homotopy category of StMod_{kP}*. Additionally, choose an equivalence*

$$\lambda_{P<Q} : \ \operatorname{res}^Q_P M_Q \to M_P$$

in StMod_{kP} *for every* $P < Q$*. We insist that both of these sets of choices be invariant under conjugation.*

For any $g \in N_G(P < Q)$ *we obtain a potentially non-commutative diagram*

$$\begin{array}{ccc} g \otimes_P M_Q & \xrightarrow{\lambda^Q_g} & M_Q \\ {\scriptstyle g\otimes_P \lambda_{P<Q}}\downarrow & & \downarrow {\scriptstyle \lambda_{P<Q}} \\ g \otimes_P M_P & \xrightarrow{\lambda^P_g} & M_P \end{array}$$

in the homotopy category of StMod_{kP}*. We define a cocycle* $\beta_M \in \mathrm{C}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times))$ *by letting* $\beta_M(P < Q)(g)$ *be the element in* $k^\times$ *corresponding to the automorphism of* M_P *given by the above diagram. The class* $\beta\big((M_P)\big)$ *is represented by* β_M*.*

5.13. **Notation.** When the G-stable endotrivial S-module (M_P) is clear from context, we will drop the subscript on β_M and use β to refer to both the homomorphism and the cocycle.

5.14. *Remark.* The condition in Diagram 5.12 is necessary to ensure that $\beta_M(P < Q)$ is actually a character. It can always be satisfied: this is exactly what we proved in Proposition 5.9 in order to show that γ lies in $\mathrm{H}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times))$.

The conjugation invariance condition that we imposed on the λ^P_g and $\lambda_{P<Q}$ is necessary to ensure that β_M is a Bredon 1-cochain, *i.e.* lies in the G-fixed points in (5.2). It can always be satisfied by choosing such data for representatives of conjugacy classes of 0- and 1-simplices in $\mathscr{C}$ and extending by conjugation.

5.15. *Remark.* Note that in many cases we have to specify significantly less data in order to compute β than we do to compute α: for the latter we need to choose

$$\lambda_g : \ \operatorname{res}_g M_Q \to M_P$$

for every $g : G/P \to G/Q$ in $\mathcal{O}_\mathcal{S}(G)$, whereas for the former we need only choose such an equivalence when g is an isomorphism in $\mathcal{T}_\mathcal{S}(G)$ or an inclusion $P \hookrightarrow Q$. Furthermore, if it happens that $\mathscr{C}/G$ is isomorphic to a subposet $\mathscr{C}_0 \subseteq \mathscr{C}$ then we only need to specify λ_g when g is an automorphism of a subgroup in $\mathscr{C}_0$ or an inclusion of subgroups in $\mathscr{C}_0$, which again makes computations much simpler in practice. Examples where this occurs include the case $G = \mathrm{PSL}_3(p)$ with $p \equiv 1 \bmod 3$ and the case $G = \Sigma_n$ with $p^2 \leq n < p^2 + p$; see Sections 10 and 12 respectively.

Proof of Proposition 5.11. This is a computation working through the construction of the isotropy spectral sequence. We first extend our choices of λ^P_g and $\lambda_{P<Q}$ to the rest of the transport category, *i.e.* we make choices of equivalences

$$\lambda_g : \ g \otimes_Q M_Q \to M_P \tag{5.16}$$

in StMod_{kP} for every remaining $g^{-1} : P \to Q$ in $\mathcal{T}_\mathcal{S}(G)$. As in Remark 5.7, these choices determine a cocyle $\gamma \in \mathrm{C}^2(\mathcal{T}_\mathcal{S}(G); k^\times)$ that represents the cohomology class of the image of α in $\mathrm{H}^2(\mathcal{T}_\mathcal{S}(G); k^\times)$.

We will now check that the class represented by γ in $\mathrm{H}^2(\mathcal{T}_\mathcal{S}(G); k^\times)$ restricts to the class of the cocycle β defined in the statement of the proposition. Let

$$F_r^n := \bigoplus_{\substack{s+t=n \\ s\geq r}} \mathrm{Hom}_G(\mathrm{C}_t(\mathrm{E}G), \mathrm{C}^s(\mathscr{C}; k^\times))$$

be the filtration $F_0^\bullet \supseteq F_1^\bullet \supseteq \ldots$ that gives rise to the isotropy spectral sequence. The inclusion

$$\mathrm{H}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times)) \to \mathrm{H}^2(\mathcal{T}_\mathcal{S}(G); k^\times)$$

of Lemma 5.5 is induced by the composition

$$\mathrm{C}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times)) \cong \mathrm{H}^1(G; \mathrm{C}^1(\mathscr{C}; k^\times)) \cong \mathrm{H}^2(F_1^\bullet/F_2^\bullet) \leftarrow \mathrm{H}^2(F_1^\bullet) \to \mathrm{H}^2(F_0^\bullet) \xleftarrow{\cong} \mathrm{H}^2(\mathcal{T}_\mathcal{S}(G); k^\times),$$

where a left-facing arrow means choosing a lift along that map (the image is independent of the choice of lift).

The right-most isomorphism

$$\mathrm{H}^2(\mathcal{T}_\mathcal{S}(G); k^\times) \xrightarrow{\cong} \mathrm{H}^2(F_0^\bullet) = \mathrm{H}^2(\mathrm{Tot}^\bullet \mathrm{Hom}_G(\mathrm{C}_\bullet(\mathrm{E}G), \mathrm{C}^\bullet(\mathscr{C}; k^\times))$$

is given by pulling back along some choice of chain homotopy inverse to the Alexander–Whitney map, then applying the tensor-hom adjunction. Under one such choice of isomorphism, γ corresponds to the element of $\mathrm{Tot}^2 \mathrm{Hom}_G(\mathrm{C}_\bullet(\mathrm{E}G), \mathrm{C}^\bullet(\mathscr{C}; k^\times))$ that sends

$$\begin{aligned} 1 &\quad\mapsto\quad [P \leq Q \leq R \;\mapsto\; \alpha(P \leq Q \leq R)] \\ 1 \xrightarrow{g} g &\quad\mapsto\quad \left[P \leq Q \mapsto \frac{\alpha(P \leq Q \xrightarrow{g} {}^gQ)}{\alpha(P \xrightarrow{g} {}^gP \leq {}^gQ)} \right] \\ 1 \xrightarrow{g} g \xrightarrow{h} hg &\quad\mapsto\quad [P \mapsto \alpha(P \xrightarrow{g} {}^gP \xrightarrow{h} {}^{hg}P)]. \end{aligned} \tag{5.17}$$

The component of γ labelled (5.17), living in $\mathrm{Hom}_G(\mathrm{C}_2(\mathrm{E}G), \mathrm{C}^0(\mathscr{C}; k^\times))$, is equivalent to zero because we chose the λ_g^P to make Diagram 5.12 commute. (More precisely, the subcategory of the transport category on conjugates of P is a connected groupoid, and hence is equivalent to the full subcategory on the object P. We chose the λ_g^P compatibly on P, so when we extended these choices at the start of the proof, we can do so in such a way that γ is equal to zero.) Therefore we can lift γ to the class in $\mathrm{H}^2(F_1^\bullet)$ that sends

$$\begin{aligned} 1 &\quad\mapsto\quad [P \leq Q \leq R \;\mapsto\; \alpha(P \leq Q \leq R)] \\ 1 \xrightarrow{g} g &\quad\mapsto\quad \left[P \leq Q \mapsto \frac{\alpha(P \leq Q \xrightarrow{g} {}^gQ)}{\alpha(P \xrightarrow{g} {}^gP \leq {}^gQ)} \right]. \end{aligned} \tag{5.18}$$

Under the map $\mathrm{H}^2(F_1^\bullet) \to \mathrm{H}^2(F_1^\bullet/F_2^\bullet) \cong \mathrm{H}^1(G; \mathrm{C}^1(\mathscr{C}; k^\times))$, this class is sent to the homomorphism labelled (5.18). By comparing to the formula of α given in Theorem 4.5, we see that this homomorphism represents a class that identifies with β under the isomorphism

$$\mathrm{H}^1(G; \mathrm{C}^1(\mathscr{C}; k^\times)) \cong \mathrm{C}^1_G(\mathscr{C}; \mathrm{H}^1(-; k^\times)),$$

thereby completing the proof. □

6. Compatible actions in the homotopy category

The obstruction β in the Bredon cohomology group $\mathrm{H}^1_G(\mathscr{C};\, \mathrm{H}^1(-;\, k^\times))$ demonstrates a connection between lifting endotrivial modules and one-dimensional characters: the coefficients of the obstruction group take a subgroup P to the group of one-dimensional characters of $N_G(P)$. (The normaliser of P appears here because this is the stabiliser in G of P, considered as a 0-simplex of the G-space $\mathscr{C}$.) In the next section, we will expand on this connection, giving a necessary and sufficient condition for a G-stable endotrivial S-module to lift that is completely algebraic and makes no reference to obstruction groups; the condition involves only specifying one-dimensional characters of $N_G(P)$ for varying P. The current section lays the groundwork for this by analysing the relationship between the definition of β and one-dimensional characters. Throughout this section, we work in the stable module category (*i.e.* the homotopy category of the stable module ∞-category) unless stated otherwise.

We start by giving a name to a concept that has already appeared several times in Section 5. It captures the part of the structure of an $N_G(P)$-module that can still be seen in StMod_{kP}. Closely related objects were considered in [Bal15, Theorem 9.9].

6.1. **Definition.** Let N be a finite group with a normal p-subgroup P and let $M \in \mathrm{StMod}_{kP}$. We say M is *N-stable* if $g \otimes_P M \simeq M$ for every $g \in N$. A *compatible N-action* on an N-stable module M is a set of equivalences

$$\{\, \lambda_g :\ g \otimes_P M \xrightarrow{\sim} M \mid g \in N \,\}$$

such that

$$\begin{array}{ccc} gh \otimes_P M & \xrightarrow{\ g \otimes_P \lambda_h\ } & g \otimes_P M \\ & {\scriptstyle \lambda_{gh}} \searrow \quad \swarrow {\scriptstyle \lambda_g} & \\ & M & \end{array} \tag{6.2}$$

commutes in the homotopy category of StMod_{kP} for every $g, h \in N$. A map $f : M \to M'$ in StMod_{kP} between modules with compatible N-actions is *N-equivariant* if it respects the N-actions.

6.3. *Remark.* We have been using the term "G-stable endotrivial S-module" to mean an element of $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$, *i.e.* an element of $T(S)$ that is invariant under restriction and conjugation by elements of G. Since P is normal in N, if M is an N-stable P-module in the sense of Definition 6.1 that is also endotrivial, then M is an element of $\lim_{\mathcal{O}_{\mathcal{S}}(N)^{\mathrm{op}}_{\leq P}} T(-)$, where $\mathcal{O}_{\mathcal{S}}(N)_{\leq P}$ denotes the full subcategory of $\mathcal{O}_{\mathcal{S}}(N)$ on objects G/Q with $Q \leq P$.

6.4. *Remark.* Let $i : P \to N$ be the inclusion exhibiting P as a normal subgroup of N and consider the induction/restriction monad $i^* i_!$ on StMod_{kP}. By normality, the underlying functor of this monad can be identified with $k[N/P] \otimes_k (-)$. The resulting algebra structure on $k[N/P]$ is then determined by the formula $[g] \cdot [h] = [gh]$ for cosets $[g], [h] \in N/P$.

A compatible N-action on $M \in \mathrm{StMod}_{kP}$ is the same data as an action of $k[N/P]$ on M in the homotopy category of StMod_{kP}. We could consequently construct an ∞-categorical version of the above definition, namely the ∞-category of algebras over the monad $i^* i_!$ in StMod_{kP}:

$$\mathrm{Alg}_{\mathrm{StMod}_{kP}}(i^* i_!) \simeq \mathrm{Mod}_{\mathrm{StMod}_{kP}}(k[N/P]).$$

However, for our purposes it is sufficient to work with the hands-on definition given above.

Compatible actions are more rigid than they appear at first sight, and the next lemma shows that their behaviour is controlled on subgroups of P:

6.5. **Lemma.** *Let N be a finite group with a normal p-subgroup P, and let $Z \leq P$ be a subgroup that is normal in N. Let $M \in \mathrm{StMod}_{kP}$ be an N-stable endotrivial module. The group of one-dimensional characters $\mathrm{H}^1(N;\, k^\times)$ acts freely and transitively on the set of compatible N-actions on M, and restriction induces an $\mathrm{H}^1(N;\, k^\times)$-equivariant bijection*

$$\{\, \textit{compatible } N\textit{-actions on } M \,\} \xrightarrow{\cong} \{\, \textit{compatible } N\textit{-actions on } \mathrm{res}^P_Z M \,\}.$$

Here we consider two compatible actions λ_g and λ'_g on M to be the same if id_M is N-equivariant, i.e. *if λ_g is homotopic to λ'_g for every $g \in N$.*

Proof. Let $\varphi \in \mathrm{H}^1(N;\, k^\times)$ act on the set of compatible N-actions on M by sending λ_g to $\varphi(g)\lambda_g$. By Lemma 2.7, this action is free and transitive. It is clear that a compatible N-action on M gives rise to one on $\mathrm{res}^P_Z M$ by restriction, so we need to provide the inverse map. Suppose that $\{\lambda_g : g \otimes_Z \mathrm{res}^P_Z M \to \mathrm{res}^P_Z M\}$ is a compatible N-action. Since $g \otimes_P M$ and M are equivalent endotrivial P-modules, restriction induces an isomorphism of one-dimensional k-vector spaces

$$\mathrm{res}^P_Z : \pi_0 \underline{\mathrm{Hom}}_P(g \otimes_P M, M) \xrightarrow{\cong} \pi_0 \underline{\mathrm{Hom}}_Z(g \otimes_Z \mathrm{res}^P_Z M, \mathrm{res}^P_Z M).$$

We define $\mu_g : g \otimes_P M \to M$ to be the unique (up to homotopy) map that restricts to λ_g. Since the λ_g make Diagram 6.2 commute, so too do the μ_g. □

6.6. *Remark.* Any compatible N-action on an endotrivial P-module M necessarily has the property that for every $x \in P$, the map $\lambda_x : x \otimes_P M \to M$ is homotopic to the natural map given by the P-module structure of M. Since P is a p-group, the group $\mathrm{H}^1(P;\, k^\times)$ is trivial. Lemma 6.5 therefore implies that all compatible actions of P on M are homotopic, and the P-module structure of M is one such compatible action.

6.7. *Remark.* Let M and M' be endotrivial P-modules with compatible N-actions. Since they are endotrivial, either all equivalences $M \xrightarrow{\sim} M'$ as stable P-modules are N-equivariant or none of them are.

Similarly, suppose that M is an endotrivial P-module with a compatible N-action and that M' is a P-module equivalent to M in StMod_{kP}. There is a unique compatible N-action on M' induced by the existence of an equivalence $M \simeq M'$, *i.e.* the induced N-action does not depend on the choice of equivalence.

The Heller shift Ω on StMod_{kP} induces a shift functor $\tilde{\Omega}$ on stable P-modules with compatible N-actions: $\tilde{\Omega}M$ has ΩM as its underlying stable P-module and a compatible N-action given by

$$g \otimes_P \Omega M \simeq \Omega(g \otimes_P M) \xrightarrow{\Omega\lambda_g} \Omega M.$$

When P is cyclic, the periodicity of $\tilde{\Omega}$ may differ from that of Ω, since the N-action is altered by the functor $\tilde{\Omega}$. To understand this difference in periodicity, we need to determine the action of $N_G(P)$ on Tate cohomology.

6.8. **Lemma.** *Let Z be a non-trivial cyclic p-group that is normal in a finite group N. Let M be a stable Z-module with a compatible N-action such that $M \simeq \Omega^n k$ as stable Z-modules. Let l denote the periodicity of a projective resolution of k as a Z-module (*i.e. *$l = 1$ if Z is cyclic of order two and $l = 2$ otherwise). There is a natural isomorphism*

$$\hat{\mathrm{H}}^l(Z;\, k) \otimes \hat{\mathrm{H}}^n(Z;\, M) \xrightarrow{\cong} \hat{\mathrm{H}}^{n+l}(Z;\, M)$$

as one-dimensional characters of N, which is induced by composition of morphisms in the stable module category of Z.

Proof. Since $M \simeq \Omega^n k$ as stable Z-modules, both sides of the morphism induced by composition are one-dimensional k-vector spaces. The interesting part of the statement is that composition is N-equivariant.

We will work in the homotopy category of the stable module ∞-category of Z, and temporarily introduce the notation $[M, N]$ for the morphism set $\pi_0 \underline{\mathrm{Hom}}_Z(M, N)$ in that homotopy category. Recall that $\hat{\mathrm{H}}^n(Z;\, M) \cong [\Omega^n k, M]$. The action of $g \in N$ on $\hat{\mathrm{H}}^n(Z;\, M)$, considered as a k-vector space, is given by

$$g \cdot - :\ [\Omega^n k, M] \xrightarrow{g\otimes_Z -} [g \otimes_Z \Omega^n k, g \otimes_Z M] \to [\Omega^n k, g \otimes_Z M] \to [\Omega^n k, M],$$

where the middle map is induced by the the natural equivalence $g \otimes_Z \Omega^n k \simeq \Omega^n(g \otimes_Z k) \simeq \Omega^n k$ and the last map by composing with the action $g \otimes_Z M \to M$ of g on M. It suffices to show that each of these maps respects composition

$$[\Omega^l k, k] \otimes [\Omega^n k, M] \xrightarrow{\Omega^n \otimes \mathrm{id}} [\Omega^{n+l} k, \Omega^n k] \otimes [\Omega^n k, M] \xrightarrow{\circ} [\Omega^{n+l} k, M].$$

This is a straightforward check using the fact that $g \otimes_Z -$ commutes with Ω. □

6.9. **Lemma.** *Let Z be a non-trivial cyclic p-group that is normal in a finite group N.*

When p is odd, let Y be the unique subgroup of Z of order p (which will also be normal in N), and let ν denote the twisting character

$$\nu :\ N \to \mathrm{Aut}(Y) \cong \mathbb{F}_p^\times \leq k^\times$$

given by the right action of N on Y. We have

$$\hat{\mathrm{H}}^2(Z;\, k) = \nu$$

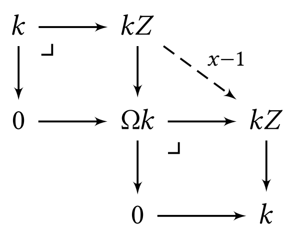
as one-dimensional characters of N.

*When $p = 2$, let l denote the periodicity of a projective resolution of k as a Z-module (*i.e. $l = 1$ *if Z is cyclic of order two and $l = 2$ otherwise). The one-dimensional character $\hat{\mathrm{H}}^l(Z;\, k)$ is trivial.*

Proof. We deal first with the case where p is odd. Since $k \simeq \Omega^2 k$, the Tate cohomology $\hat{\mathrm{H}}^2(Z;\, k)$ is one-dimensional, and we wish to determine the N-action on it. Let $[M, N]$ again denote $\pi_0 \underline{\mathrm{Hom}}_Z(M, N)$. The action of $g \in N$ on $\hat{\mathrm{H}}^2(Z;\, k)$ is given by

$$g \cdot - :\ [\Omega^2 k, k] \xrightarrow{g\otimes_Z -} [g \otimes_Z \Omega^2 k, g \otimes_Z k] \cong [\Omega^2 k, k]. \tag{6.10}$$

We will choose a convenient model for $\Omega^2 k$ and compute this action explicitly. Let x be a generator of Z. We have strict pullback squares in the model category Mod_{kZ}



that are also homotopy pullbacks, which witness that k is a model for $\Omega^2 k$. These pullbacks are completely determined by the resolution of k as a kZ-module

$$k \to kZ \xrightarrow{x-1} kZ \to k \tag{6.11}$$

that appears along the top and right edges of the above diagram.

With this identification, we can consider $\mathrm{id} : k \to k$ as an element of $\hat{\mathrm{H}}^2(Z; k)$ and compute where it is sent to by the action of g given in (6.10). The crucial point here is that in the isomorphism

$$[g \otimes_Z k, g \otimes_Z k] \cong [k, k],$$

the equivalence between $g \otimes_Z k$ and k is different in the domain and codomain: in the domain, we are treating k as a model for $\Omega^2 k$ and so need to take the data of the resolution (6.11) into account, whereas in the codomain the equivalence is induced by the (trivial) action of g on k. Therefore, the action of g on $\hat{\mathrm{H}}^2(Z; k)$ is given by multiplication by the scalar λ that makes the following diagram commute in Mod_{kZ}:

$$\begin{array}{ccccccc}
k & \longrightarrow & kZ & \xrightarrow{x-1} & kZ & \longrightarrow & k \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}} \\
k & \longrightarrow & g \otimes_Z kZ & \xrightarrow[g \otimes_Z (x-1)]{} & g \otimes_Z kZ & \longrightarrow & k
\end{array}$$

This amounts to lifting the identity to a comparison map between the two resolutions of k.

We can choose f_1 to send 1 to $g \otimes_Z 1$, *i.e.* send z to $g \otimes_Z z^g$. We need to choose f_2 such that

$$(g \otimes_Z (x-1)) \cdot f_2(1) = f_1(x-1) = g \otimes_Z (x^g - 1).$$

Write $|Z| = p^r$. Expanding with respect to the basis given by powers of $(x-1)$, we have a unique expression

$$x^g - 1 = \sum_{1 \le i < p^r} a_i (x-1)^i \tag{6.12}$$

for some scalars $a_i \in k$. We can therefore choose $f_2(1) := \sum_{1 \le i < p^r} a_i g \otimes_Z (x-1)^{i-1}$.

The map $k \to g \otimes_Z kZ$ is given by sending 1 to $g \otimes_Z N_Z$, where N_Z denotes the norm element of kZ. By definition, λ satisfies

$$\lambda g \otimes_Z N_Z = f_2(N_Z) = a_1 g \otimes_Z N_Z,$$

so $\lambda = a_1$. To determine a_1, we observe that

$$x^m = ((x-1) + 1)^m \equiv m(x-1) + 1 \bmod (x-1)^2$$

for any integer m, and hence for an appropriate choice of m we have

$$x^g - 1 = x^m - 1 \equiv m(x-1) \bmod (x-1)^2.$$

Comparing with Equation (6.12), we see that a_1, and hence also λ, is equal to the image of m in k whenever m satisfies $x^m = x^g$. We learnt the above trick of expanding modulo $(x-1)^2$ from [MT07].

Since Y is generated by $x^{p^{r-1}}$, if $x^m = x^g$ then we have

$$(x^{p^{r-1}})^{\nu(g)} := (x^{p^{r-1}})^g = (x^{p^{r-1}})^m$$

and so $\nu(g)$ is also equal to the image of m in k. Therefore, $\lambda = \nu(g)$ and $\hat{\mathrm{H}}^2(Z; k) = \nu$ as one-dimensional characters of N.

When $p = 2$ and $l = 2$, one can carry out the same computation to get $\lambda = \nu(g)$. However, in this case the character ν is trivial: the image of m in k is always 1. For $p = 2$ and $l = 1$, we have $g \otimes_{C_2} kC_2 \cong kC_2$, so the action of g on $\hat{\mathrm{H}}^1(Z; k)$ is again trivial. □

Combining these two lemmas, we obtain:

6.13. **Corollary.** *Let Z be a non-trivial cyclic p-group that is normal in a finite group N, and let ν be the twisting character defined in Lemma 6.9. Let M be a stable Z-module with a compatible N-action such that $M \simeq \Omega^n k$ as stable Z-modules. When p is odd we have*

$$\hat{\mathrm{H}}^{n+2}(Z;\, M) \cong \nu \otimes \hat{\mathrm{H}}^n(Z;\, M)$$

as one-dimensional N-modules, and when $p = 2$ we have

$$\hat{\mathrm{H}}^{n+l}(Z;\, M) \cong \hat{\mathrm{H}}^n(Z;\, M),$$

*where l is the periodicity of a projective resolution of k as a Z-module (*i.e. *$l = 1$ if Z is cyclic of order two and $l = 2$ otherwise).*

6.14. **Notation.** Let Z be a normal subgroup of N. For any $M \in \mathrm{StMod}_{kN}$, we get an induced compatible N-action on the stable Z-module $\operatorname{res}^N_Z M$. When we want to emphasise the fact that we wish to consider $\operatorname{res}^N_Z M$ with its compatible N-action, we will denote it by $\widetilde{\operatorname{res}}^N_Z M$.

We can now use Corollary 6.13 to show several useful facts about stable modules with compatible actions:

6.15. **Proposition.** *Let Z be a non-trivial p-group that is normal in a group N. In the case where Z is cyclic, let ν denote the twisting character defined in Lemma 6.9.*

(i) For any $M \in \mathrm{StMod}_{kN}$ we have an equivalence $\widetilde{\operatorname{res}}^N_Z(\Omega^n M) \simeq \tilde{\Omega}^n \widetilde{\operatorname{res}}^N_Z M$ as stable Z-modules with compatible N-actions.

(ii) If Z is abelian, then every endotrivial Z-module with a compatible N-action is equivalent to $\widetilde{\operatorname{res}}^N_Z(\Omega^n \varphi)$ for some $n \in \mathbb{Z}$ and $\varphi \in \mathrm{H}^1(N;\, k^\times)$.

(iii) Let $\varphi, \psi \in \mathrm{H}^1(N;\, k^\times)$ be one-dimensional characters. There is an equivalence

$$\widetilde{\operatorname{res}}^N_Z(\Omega^n \varphi) \simeq \widetilde{\operatorname{res}}^N_Z(\Omega^n \psi)$$

of stable Z-modules with compatible N-actions if and only if $\varphi = \psi$ as characters.

(iv) If Z is cyclic and p is odd, then $\widetilde{\operatorname{res}}^N_Z(\Omega^2 \varphi) \simeq \widetilde{\operatorname{res}}^N_Z(\nu^{-1}\varphi)$ as stable Z-modules with compatible N-actions.

(v) If Z is cyclic and $p = 2$, then $\widetilde{\operatorname{res}}^N_Z(\Omega\varphi) \simeq \widetilde{\operatorname{res}}^N_Z \varphi$ as stable Z-modules with compatible N-actions.

Proof.

(i) It is enough to prove this for $n = \pm 1$. We will prove it for $n = 1$, since the $n = -1$ case is just a dual argument. We need to show that, for any $g \in N$, the action of g on the N-module ΩM is given by

$$g \otimes_N \Omega M \simeq \Omega(g \otimes_N M) \xrightarrow{\Omega(\mathrm{act}_g)} \Omega M$$

up to homotopy. Let P_M denote the projective cover of M. We have a commutative diagram in the 1-category Mod_{kN}

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(g \otimes_N M) & \longrightarrow & g \otimes_N P_M & \longrightarrow & g \otimes_N M & \longrightarrow & 0 \\
 & & \big\downarrow \cong & & \big\| & & \big\| & & \\
0 & \longrightarrow & g \otimes_N \Omega M & \longrightarrow & g \otimes_N P_M & \longrightarrow & g \otimes_N M & \longrightarrow & 0 \\
 & & \big\downarrow \mathrm{act}_g & & \big\downarrow \mathrm{act}_g & & \big\downarrow \mathrm{act}_g & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

since the horizontal maps are N-equivariant. This shows that the left-hand composite is homotopic to $\Omega(\mathrm{act}_g)$, proving the claim.

(ii) This is a consequence of Dade's Theorem 2.8: the underlying stable Z-module must be equivalent to $\Omega^n k$ for some k, so if we apply $\tilde{\Omega}^{-n}$ then we get a compatible action on k. These naturally biject with one-dimensional characters of N.

(iii) We can apply $\tilde{\Omega}^{-n}$ to reduce to the case where $n = 0$. The underlying modules $\mathrm{res}^N_Z\,\varphi$ and $\mathrm{res}^N_Z\,\psi$ are both k, and $\widetilde{\mathrm{res}}^N_Z\varphi \simeq \widetilde{\mathrm{res}}^N_Z\psi$ if and only if the diagram

$$\begin{array}{ccc} g \otimes_N \varphi & \xrightarrow{\varphi(g)} & \varphi \\ \big\downarrow{\scriptstyle g\otimes_N f} & & \big\downarrow{\scriptstyle f} \\ g \otimes_N \psi & \xrightarrow{\psi(g)} & \psi \end{array}$$

commutes up to homotopy for all $g \in N$, which in turn is if and only if $\varphi = \psi$ as characters.

(iv) & (v) In the case where p is odd, the module $\mathrm{res}^N_Z(\Omega^2\varphi)$ is equivalent to k, and hence

$$\widetilde{\mathrm{res}}^N_Z(\Omega^2\varphi) \simeq \widetilde{\mathrm{res}}^N_Z\psi$$

for some one-dimensional character ψ. To determine ψ, we can apply zeroth Tate cohomology, and Corollary 6.13 tells us that

$$\hat{\mathrm{H}}^0(Z;\,\Omega^2\varphi) \cong \hat{\mathrm{H}}^{-2}(Z;\,\varphi) \cong \nu^{-1} \otimes \varphi.$$

A similar argument works when p is even, in which case ν is the trivial character. □

6.16. *Remark.* Despite Proposition 6.15(i), Ω and $\tilde{\Omega}$ can have quite different behaviour. In particular, even if $\tilde{\Omega}$ is periodic, it need not be the case that Ω be periodic. One example where this happens is when Z is the trivial group and N is an elementary abelian subgroup of rank at least two: then Ω is not periodic but the category of stable Z-modules with compatible N-action is trivial, so $\tilde{\Omega}$ is the identity.

A less trivial example occurs when Z is a cyclic group and N contains an elementary abelian subgroup of rank at least two: Ω is again not periodic, but the periodicity of $\tilde{\Omega}$ is at most twice the order of ν_Z in $\mathrm{H}^1(N;\,k^\times)$.

7. LIFTING VIA ORIENTATIONS

In this section, we give a more explicit condition for lifting a G-stable endotrivial S-module $(M_P)_{G/P\in\mathcal{O}_S(G)}$. For a suitable collection $\mathscr{C}$, such as $\mathcal{A}_p(G)$, the module M_S lifts to $T(G)$ if and only if we can specify a one-dimensional character of $N_G(P)$ for every $P \in \mathscr{C}$ such that the characters satisfy the compatibility condition stated in Proposition 7.5. We will refer to such a set of compatible characters as an *orientation*. This reduces the question of lifting G-stable endotrivial modules to local group theory, since we just need to understand the one-dimensional characters of normalisers of p-subgroups and how they behave under restriction.

Viewing the obstruction to lifting in terms of characters makes clear exactly "what can go wrong" when trying to lift: as explained in the introduction, when we attempt to lift a G-stable endotrivial S-module to G, we have to take into account the possibility that there is a p'-element x that acts non-trivially on a cyclic subgroup Z, in which case the order of Ωk in $T(Z \rtimes \langle x\rangle)$ is larger than its order in $T(Z)$. This puts extra compatibility conditions on type functions of modules in $T(G)$ that are not visible for modules in $\lim_{\mathcal{O}_S^{\mathrm{op}}} T(-)$.

Recall that by Dade's Theorem 2.8, for abelian p-groups, every endotrivial module is equivalent to a Heller shift of the unit k. With this in mind, we give a variation of a definition from [CMT14, §2]:

7.1. **Definition.** Let $(M_P) \in \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-)$ be a G-stable endotrivial S-module. A *type function* for (M_P) on a collection $\mathscr{C}$ is a function $n : \mathscr{C} \to \mathbb{Z}$ that is invariant under G-conjugation and that satisfies $\mathrm{res}^P_{Z(P)} M_P \simeq \Omega^{n(P)} k$ for all $G/P \in \mathcal{O}_{\mathscr{C}}(G)$.

7.2. *Remark.* All type functions for a fixed G-stable endotrivial S-module (M_P) agree when restricted to the subset of $\mathscr{C}$ consisting of subgroups with non-cyclic centre. In other words, the only flexibility one has when choosing a type function n for (M_P) is to change its value on a subgroup P with cyclic centre: if $Z(P) \cong C_2$, then there are no constraints on $n(P)$, and otherwise $n(P)$ is determined modulo two. For the same reason, the restriction that n be conjugation-invariant will automatically hold whenever $Z(P)$ is non-cyclic.

7.3. *Remark.* The definition of the type of a module in [CMT14, §2], which we will briefly recall, is closely related to our definition of a type function on $\mathcal{S}_p(G)$. When the p-rank of G is one, the type of a module is undefined. When the p-rank is two, we choose representatives $E_1, \dots, E_r$ for the G-conjugacy classes of maximal elementary abelian subgroups of G. The type of a G-module M is defined to be the r-tuple $(n_1, \dots, n_r)$, where $\mathrm{res}^G_{E_i} M \simeq \Omega^{n_i} k$. When the p-rank is greater than two, let $E_2, \dots, E_r$ be representatives for the G-conjugacy classes of rank two maximal elementary abelian subgroups of G and let E_1 be a maximal elementary abelian subgroup of rank greater than two. We define the type of M to be $(n_1, \dots, n_r)$ where the n_i are defined as before.

A type function $n : \mathcal{S}_p(G) \to \mathbb{Z}$ for (M_P) determines the type of M_S by restricting to a subset of $\mathcal{S}_p(G)$ consisting of a maximal elementary abelian subgroup of rank at least three (if G has such a subgroup) together with a representative of each G-conjugacy class of rank two maximal elementary abelian subgroups.

Conversely, the restriction homomorphism

$$T(S) \to \prod_{1 \le i \le r} T(E_i)$$

has kernel equal to the torsion subgroup of $T(S)$, so specifying the type of a G-stable endotrivial module determines it up to torsion. Therefore, in the absence of torsion, the type of a G-stable endotrivial S-module determines its type function on $\mathcal{S}_p(G)$ up to the ambiguity mentioned in Remark 7.2.

When there is torsion in $T(S)$, a type function can contain more information than the type: when S is generalised quaternion or cyclic of order at least three, there is no definition of the type of a module, yet type functions on $\mathcal{S}_p(G)$ can distinguish between Ωk and k. Similarly, when S is the semi-dihedral 2-group with presentation

$$\langle\, x, y \mid x^{2m} = y^2 = 1, yxy = x^{m-1} \,\rangle,$$

the generator $\Omega^1 L$ of the torsion part of $T(SD_{4m})$ that was constructed in [CT00, Theorem 7.1] has a non-trivial restriction to $T(\langle y \rangle)$ but has a trivial restriction to every elementary abelian subgroup. Hence type functions on $\mathcal{S}_p(G)$ can distinguish between $\Omega^1 L$ and k despite these modules having the same type.

7.4. **Definition.** Let Z be a non-trivial normal p-subgroup of a finite group N. In order to avoid the need to split into cases depending on whether or not Z is cyclic, we will make the following convention: the *twisting character ν for the right action of N on Z* is defined to be

- the trivial character of N, when Z is non-cyclic, and
- the character defined in Lemma 6.9, when Z is cyclic.

7.5. **Proposition.** *Let $\mathscr{C}$ be a collection of non-trivial p-subgroups such that $\mathscr{C} \hookrightarrow \mathcal{S}_p(G)$ is a G-homotopy equivalence and for every $P < Q$ in $\mathscr{C}$, there is an abelian p-subgroup $V \leq Q$ that is normalised by $N_G(P < Q)$ and that contains both $Z(P)$ and $Z(Q)$ as subgroups.*

Let $(M_P)_{G/P \in \mathcal{O}_S(G)}$ be a G-stable endotrivial S-module, and let $n : \mathcal{S}_p(G) \to \mathbb{Z}$ be any type function for (M_P). The module M_S lifts to an endotrivial module in $T(G)$ if and only if it is possible to specify one-dimensional characters $\{ \varphi_P \in \mathrm{H}^1(N_G(P); k^\times) : P \in \mathscr{C} \}$ satisfying the following conditions for every $P < Q$ in $\mathscr{C}$:

(i) If $p = 2$, then $\varphi_P = \varphi_Q$ as one-dimensional characters of $N_G(P < Q)$.
(ii) If p is odd, then
$$\nu_P^{(n(V)-n(P))/2} \varphi_P = \nu_Q^{(n(V)-n(Q))/2} \varphi_Q$$
as one-dimensional characters of $N_G(P < Q)$, where ν_P is the twisting character for the right action of $N_G(P)$ on $Z(P)$, as in Definition 7.4, and V is any abelian p-subgroup of Q that is normalised by $N_G(P < Q)$ and that contains both $Z(P)$ and $Z(Q)$ as subgroups.
(iii) For every $g \in G$ we have $g \otimes_{N_G(P)} \varphi_P = \varphi_{gP}$.

7.6. **Definition.** We will say that a choice of characters $\{ \varphi_P \in \mathrm{H}^1(N_G(P); k^\times) : P \in \mathscr{C} \}$ satisfying the conditions in Proposition 7.5 is an *orientation* for n.

7.7. *Remark.* When p is odd, we necessarily have $n(V) \equiv n(P) \equiv n(Q) \bmod 2$, so the exponents appearing in condition (ii) are integers. Additionally, condition (ii) does not depend on the choice of V: let W be the intersection of all abelian p-subgroups $V \leq Q$ that are normalised by $N_G(P < Q)$ and that contain both $Z(P)$ and $Z(Q)$. If W is non-cyclic, then $n(V)$ is independent of V. If W is cyclic, then the twisting characters for the right actions of N on W, on $Z(P)$, and on $Z(Q)$ all coincide.

7.8. *Remark.* We emphasise that the choice of type function $n : \mathcal{S}_p(G) \to \mathbb{Z}$ does not affect whether or not a module lifts: as in Remark 7.2, the type function is determined by (M_P) away from p-subgroups with cyclic centres. Any change in the type function on a p-subgroup with a cyclic centre can be corrected for by multiplying the orientation characters by the appropriate power of a twisting character such that condition (ii) still holds.

Proof of Proposition 7.5. Suppose that we had an endotrivial module $M \in T(G)$ that restricted to M_S. We wish to construct an orientation for the type function $n : \mathcal{S}_p(G) \to \mathbb{Z}$. For every $P \in \mathscr{C}$ we get a one-dimensional $N_G(P)$-module $\hat{\mathrm{H}}^{n(P)}(Z(P); M)$, corresponding to a one-dimensional character φ_P. We check that these satisfy the three conditions for being an orientation:

(i) If $p = 2$, then $n(P) \equiv n(V)$ modulo the periodicity of a projective resolution of k as a $Z(P)$-module, and Corollary 6.13 implies that
$$\hat{\mathrm{H}}^{n(P)}(Z(P); M) \cong \hat{\mathrm{H}}^{n(V)}(Z(P); M) \cong \hat{\mathrm{H}}^{n(V)}(V; M)$$
as $N_G(P < Q)$-modules; similarly for Q instead of P.
(ii) If p is odd, then $n(P) \equiv n(V) \bmod 2$, and Corollary 6.13 implies that
$$\nu_P^{(n(V)-n(P))/2} \otimes \hat{\mathrm{H}}^{n(P)}(Z(P); M) \cong \hat{\mathrm{H}}^{n(V)}(V; M)$$
as $N_G(P < Q)$-modules; similarly for Q instead of P.
(iii) This follows since we require type functions to be constant on conjugacy classes.

Conversely, suppose that we had an orientation $\{ \varphi_P : P \in \mathscr{C} \}$ for (M_P). We wish to show that we can lift M_S to $T(G)$. By the description of the obstruction β in Proposition 5.11, it is enough to construct a compatible $N_G(P)$-action on M_P for every $P \in \mathscr{C}$, invariant under conjugation in G, such that some (and hence every) equivalence $\mathrm{res}^Q_P M_Q \to M_P$ is $N_G(P < Q)$-equivariant.

The orientation gives us a stable $Z(P)$-module with compatible $N_G(P)$ action, namely

$$L_P := \widetilde{\mathrm{res}}^{N(P)}_{Z(P)}(\Omega^{n(P)}\varphi_P).$$

Since $\mathrm{res}^P_{Z(P)} M_P \simeq L_P$ as stable $Z(P)$-modules, we get an induced compatible $N_G(P)$-action on $\mathrm{res}^P_{Z(P)} M_P$, and hence also one on M_P by Lemma 6.5. Recall from Remark 6.7 that the induced $N_G(P)$-action on $\mathrm{res}^P_{Z(P)} M_P$ is independent of the choice of equivalence as $Z(P)$-modules, because M_P is endotrivial. Condition (iii) in the statement of the proposition implies that the resulting $N_G(P)$-actions are invariant under conjugation, as required.

Let $P < Q$ be subgroups in $\mathscr{C}$, and write $N := N_G(P < Q)$. In order to complete the proof of the proposition, we check that the conditions imposed on φ_P and φ_Q imply that $M_P \simeq M_Q$ as stable P-modules with a compatible N-action. By Lemma 6.5 again, it is enough to check that they are equivalent as stable $Z(P)$-modules with a compatible N-action. By assumption, there is a subgroup $V \leq Q$ that is normalised by N and for which there exists a zig-zag $Z(P) \leq V \geq Z(Q)$. We will use this zig-zag to compare the N-actions on L_P and L_Q (and thereby the N-actions on M_P and M_Q). Note that N normalises all three subgroups of the zig-zag, so it does make sense to talk about compatible N-actions for these subgroups.

We deal first with the case where p is odd. Using Lemma 6.5 applied to the V-module $\Omega^{n(V)}k$, we see that it has a unique compatible N-action that restricts to $\widetilde{\mathrm{res}}^N_{Z(Q)} L_Q$. We will denote this stable V-module with its compatible N-action by L_V. The N-action on M_Q agrees with the N-action on L_V: both were defined such that they restrict to L_Q, and $Z(Q) \leq V \leq Q$ by assumption. Therefore, the restriction of M_Q to a stable $Z(P)$-module is equivalent to $\widetilde{\mathrm{res}}^N_{Z(P)} L_V$. By Proposition 6.15, we must have

$$L_V \simeq \widetilde{\mathrm{res}}^N_V \Omega^{n(V)} \left(\nu_Q^{(n(V)-n(Q))/2} \cdot \varphi_Q\right)$$

and on restriction to $Z(P)$ this is equivalent to

$$\widetilde{\mathrm{res}}^N_{Z(P)} \Omega^{n(P)} \left(\nu_P^{(n(P)-n(V))/2} \cdot \nu_Q^{(n(V)-n(Q))/2} \cdot \varphi_Q\right).$$

By condition (ii), this is in turn equivalent to L_P and we are done.

Finally, we deal with the case where p is even. This is identical to the previous case, except that the twisting character is always trivial and we do not necessarily have that $n(P) \equiv n(V) \bmod 2$. As before, we deduce that

$$L_V \simeq \widetilde{\mathrm{res}}^N_V \Omega^{n(V)} \varphi_Q$$

and consequently that the restriction of M_Q to a stable $Z(P)$-module with a compatible N-action is equivalent to L_P. □

7.9. *Remark.* Let $M \in T(G)$ and let $P \in \mathscr{C}$. In the above proof of Proposition 7.5, we defined $\varphi_P := \hat{\mathrm{H}}^{n(P)}(Z(P);\, M)$. We could equally well have defined φ_P to be the unique one-dimensional character such that

$$\widetilde{\mathrm{res}}^{N_G(P)}_{Z(P)} \mathrm{res}^G_{N_G(P)} M \simeq \widetilde{\mathrm{res}}^{N_G(P)}_{Z(P)}(\Omega^{n(P)}\varphi_P).$$

7.10. *Remark.* One can prove an analogous proposition with a slightly different condition on $\mathscr{C}$: instead of asking that $Z(P) \leq V \geq Z(Q)$, we could instead ask that V be non-trivial and $Z(P) \geq V \leq Z(Q)$. The proof of the proposition goes through essentially unchanged.

7.11. *Example.* Suppose that $\mathscr{C} = \mathcal{A}_p(G)$ and p is odd. Since $Z(P) = P$ for every $P \in \mathcal{A}_p(G)$, choosing $V := Q$ shows that $\mathcal{A}_p(G)$ satisfies the conditions on $\mathscr{C}$ in Proposition 7.5. The compatibility conditions on $\{ \varphi_P : P \in \mathscr{C} \}$ reduce to:

(1) if P is not cyclic, then φ_P and φ_Q agree on restriction to $N_G(P < Q)$,
(2) if P is cyclic, then
$$\varphi_P = \nu_P^{(n(P)-n(Q))/2} \cdot \varphi_Q$$
as $N_G(P < Q)$-characters, where ν_P is the twisting character induced by the right action of $N_G(P)$ on P, and
(3) for every $g \in G$, we have $g \otimes_{N_G(P)} \varphi_P = \varphi_{gP}$.

In order to be more explicit, we have split condition (ii) into two conditions here: since $n(P) = n(Q)$ when P is not cyclic, the two formulations are equivalent.

7.12. **Corollary.** *When $p = 2$, any G-stable endotrivial S-module lifts to a module in $T(G)$.*

Proof. We apply Proposition 7.5 to $\mathscr{C} = \mathcal{A}_p(G)$, and observe that letting each φ_P be the trivial character gives an orientation for any type function. □

8. An algebraic description of the group of endotrivial modules

In this section, we use the obstruction β of Section 5 to provide an algebraic description of the group of endotrivial modules of G along the lines of the viewpoint from [Gro18].

Recall that, by Remark 7.2, a G-stable endotrivial S-module $(M_P)_{G/P \in \mathcal{O}_S(G)}$ together with a type function for (M_P) on $\mathcal{A}_p(G)$ uniquely determines an extension of that type function to $\mathcal{S}_p(G)$. We will implicitly identify (M_P) with the module M_S on the Sylow subgroup, from which we can recover the other modules by restriction.

8.1. **Definition.** We define an equivalence relation $\sim$ on the set of tuples $(M_S, n, \{\varphi_V\})$, where M_S is a G-stable endotrivial S-module, $n : \mathcal{A}_p(G) \to \mathbb{Z}$ is a type function for M_S on $\mathcal{A}_p(G)$, and $\{ \varphi_V : V \in \mathcal{A}_p(G) \}$ is an orientation for the unique extension of n to $\mathcal{S}_p(G)$. We say that $(M_S, n, \{\varphi_V\}) \sim (M'_S, n', \{\varphi'_V\})$ if $M_S \simeq M'_S$ and for every $V \in \mathcal{A}_p(G)$ we have
$$\varphi_V = \begin{cases} \varphi'_V & \text{if } p = 2 \text{ or} \\ \varphi'_V \cdot \nu_V^{(n(V)-n'(V))/2} & \text{if } p \text{ odd,} \end{cases}$$
where ν_V denotes the twisting character for the right action of $N_G(V)$ on V. (Recall from Definition 7.4 that ν_V is trivial by convention when V is not cyclic.) We define $A(G)$ to be the set of tuples $(M_S, n, \{\varphi_V\})$ up to the equivalence relation $\sim$. Note that for a given equivalence class of $\sim$, the choice of type function determines the orientation; that is, there is a bijection between type functions of M_S and elements of the equivalence class.

8.2. *Remark.* It is possible to remove the equivalence relation from the definition of $A(G)$ if one is willing to put a restriction on n, for example that $0 \leq n(Z) < |T(Z)|$ for every cyclic $Z \in \mathcal{A}_p(G)$. This picks out a distinguished representative in each equivalence class. Alternatively, one could omit the type function altogether and instead provide an infinite sequence of characters, one for each possible value of $n(Z)$.

We can put an abelian group structure on $A(G)$ by defining
$$(M_S,\, n,\, \{\varphi_V\}) \cdot (M'_S,\, n',\, \{\varphi'_V\}) := (M_S \otimes M'_S,\, n + n',\, \{\varphi_V \cdot \varphi'_V\}).$$
This multiplication respects the equivalence relation, and $\{\varphi_V \cdot \varphi'_V\}$ is an orientation for $n + n'$ because both $\{\varphi_V\}$ and $\{\varphi'_V\}$ are orientations for their respective type functions.

8.3. **Theorem.** *There is an isomorphism of groups $\Theta \colon T(G) \to A(G)$ that sends an endotrivial G-module M to the equivalence class of the tuple*

$$(\operatorname{res}^G_S M,\ n,\ \varphi_V \coloneqq \hat{\mathrm{H}}^{n(V)}(V;\ M)),$$

where n is any type function $n \colon \mathcal{A}_p(G) \to \mathbb{Z}$ for M.

Proof. We compare the exact sequence in Corollary 5.10 to a similar sequence containing $A(G)$, which we will show to be exact below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(G,S) & \longrightarrow & T(G) & \longrightarrow & \lim_{\mathcal{O}_S^{\mathrm{op}}} T(-) & \xrightarrow{\beta} & \mathrm{H}^1_G(\mathcal{A}_p(G);\ \mathrm{H}^1(-;\ k^\times)) \\
 & & \downarrow{\cong} & & \downarrow{\Theta} & & \| & & \| \\
0 & \longrightarrow & \mathrm{H}^0_G(\mathcal{A}_p(G);\ \mathrm{H}^1(-;\ k^\times)) & \xrightarrow{f_0} & A(G) & \xrightarrow{f_1} & \lim_{\mathcal{O}_S^{\mathrm{op}}} T(-) & \xrightarrow{\beta} & \mathrm{H}^1_G(\mathcal{A}_p(G);\ \mathrm{H}^1(-;\ k^\times)).
\end{array}$$

The leftmost vertical map is the description of $T(G,S)$ given in [Gro18, §5]. The map f_0 is the inclusion map that sends a Bredon 0-cocycle $\{\,\varphi_V \,:\, V \in \mathcal{A}_p(G)\,\}$ to the tuple $(k, z, \{\varphi_V\})$, where z is the constant function on $\mathcal{A}_p(G)$ with value 0. The characters $\{\varphi_V\}$ define an orientation for n because, for any $V < W$, the characters φ_V and φ_W are compatible upon restriction to $N_G(V < W)$, by definition of Bredon cohomology. The map f_1 is the projection map onto the first factor, remembering only the G-stable endotrivial S-module. The only square whose commutativity needs justification is the leftmost one. This amounts to checking that the map

$$T(G,S) \to \mathrm{H}^0_G(\mathcal{A}_p(G);\ \mathrm{H}^1(-;\ k^\times))$$

is given by sending a Sylow-trivial module M to the cocycle whose value on a subgroup V is the one-dimensional $N_G(V)$-character given by $\hat{\mathrm{H}}^0(V;\ M)$. This is implicit in [Gro18], combining the description of the isomorphism in Theorem A with the proof of Proposition 5.3 (dualised to cohomology).

We now justify the exactness of the lower sequence; the theorem will then follow from the 5-lemma. The injectivity of f_0 is clear: the tuple $(k, z, \{\varphi_V\})$ represents the zero class only when all φ_V are trivial. Exactness at $A(G)$ is also straightforward: if $f_1(M_S, n, \{\varphi_V\}) \simeq k$ then by changing the representative of the equivalence class in $A(G)$, we can take n to be equal to z, and hence $\{\varphi_V\}$ forms a Bredon 0-cocycle. Finally, we consider exactness at the limit term. That $\beta f_1 = 0$ follows from Proposition 7.5: we asked that $\{\varphi_V\}$ be an orientation for n, and so $\beta(M_S) = 0$. Conversely, if $\beta(M_S) = 0$, we know that we can lift M_S to some $M \in T(G)$, by exactness of the top sequence. Applying Θ to M gives an equivalence class in $A(G)$ that lifts M_S. □

8.4. *Remark.* If $T(S)$ does have torsion, then one might worry that it could be difficult in practice to verify whether a given module in $T(S)$ is G-stable, since the type does not detect torsion. Fortunately, in this case [MT07, Theorem 3.6] and [CMT13, Theorems 4.5 and 6.4] show that $T(G) \to T(S)$ is surjective, and hence that all endotrivial S-modules are G-stable.

In the case where p is odd, we do not need to specify the module M_S in $A(G)$, because this is recoverable from the type function n. In order to make this simplification, we need to be able to recognise which type functions are realisable by a G-stable endotrivial S-module. This is the content of Proposition 8.5 below, which we state after recalling some notation from [CMT14, §3].

Let p be odd. Without loss of generality, we can assume that the torsion-free rank of $T(S)$ is at least two: otherwise, $T(S)$ is cyclic (and so detected by n) or trivial. This implies that the p-rank of G is at least two. If the p-rank of G is two, then let $E_1, \dots, E_r$ be representatives for the S-conjugacy classes of rank two maximal elementary abelian subgroups of S. If the p-rank of G is three or greater, then let $E_2, \dots, E_r$ be representatives for the S-conjugacy classes of rank two maximal elementary abelian subgroups of S, and let E_1 be a maximal elementary abelian subgroup of rank greater than two. As in [CMT14, Theorem 4.1], $T(S)$ is generated by Ωk and modules $N_2, \dots, N_r$, where each N_i restricts trivially to E_j unless $j = i$ and restricts to $\Omega^{2p}k$ on E_i.

8.5. **Proposition.** *Let p be odd, and consider the set of pairs $(n, \{\varphi_V : V \in \mathcal{A}_p(G)\})$ where $n : \mathcal{A}_p(G) \to \mathbb{Z}$, each $\varphi_V \in \mathrm{H}^1(N_G(V); k^\times)$, and n and $\{\varphi_V\}$ satisfy the following conditions:*

(i) For every $V < W$ in $\mathcal{A}_p(G)$ with V non-cyclic, we have $n(V) = n(W)$.
(ii) For every $V < W$ in $\mathcal{A}_p(G)$ with V cyclic, we have $n(V) \equiv n(W) \bmod 2$.
(iii) For every i with $1 < i \leq r$, we have $n(E_i) \equiv n(E_1) \bmod 2p$.
(iv) n is constant on G-conjugacy classes.
(v) For every $V < W$ in $\mathcal{A}_p(G)$, we have

$$\varphi_W = \nu_V^{(n(W)-n(V))/2} \varphi_V$$

as one-dimensional characters of $N_G(V < W)$, where ν_V denotes the twisting character for the right action of $N_G(V)$ on V as in Definition 7.4.

We put an equivalence relation on such pairs by defining $(n, \{\varphi_V\}) \sim (n', \{\varphi'_V\})$ if for every non-cyclic $V \in \mathcal{A}_p(G)$, we have $n(V) = n'(V)$, and for every $V \in \mathcal{A}_p(G)$ we have

$$\varphi_V = \varphi'_V \cdot \nu_V^{(n(V)-n'(V))/2}.$$

Let $B(G)$ denote the set of equivalence classes for $\sim$, and give $B(G)$ the abelian group structure

$$(n, \{\varphi_V\}) \cdot (n', \{\varphi'_V\}) := (n + n', \{\varphi_V \cdot \varphi'_V\}).$$

There is an isomorphism of groups $A(G) \to B(G)$ induced by forgetting the M_S factor.

Proof. From the description of $A(G)$ given in Theorem 8.3, it is enough to show that a function $n : \mathcal{A}_p(G) \to \mathbb{Z}$ is the type function of a G-stable endotrivial S-module if and only if n satisfies conditions (i)–(iv) in the statement of the proposition, in which case there is a unique G-stable endotrivial S-module whose type function is n.

We start with the uniqueness statement: for a p-group P, if P is cyclic or $T(P)$ has no torsion, then the restriction

$$T(P) \to \prod_{V \in \mathcal{A}_p(P)} T(V) \tag{8.6}$$

is injective. (In the cyclic case, we are using the assumption that p is odd, and otherwise the injectivity is [Pui90, Theorem 2.2].) Therefore, for any type function n, there can be at most one G-stable endotrivial S-module that realises it.

Using [CMT14, Theorem 4.1], we see that the type function of any endotrivial S-module must satisfy conditions (i)–(iii) when $W \leq S$. If (M_P) is G-stable, then condition (iv) follows from the equivalences

$$\Omega^{n({}^gV)}k \simeq M_{{}^gV} \simeq g \otimes_V M_V \simeq g \otimes_V \Omega^{n(V)}k,$$

and this implies conditions (i)–(iii) for all $V < W$ in $\mathcal{A}_p(G)$.

Conversely, given a function n satisfying (i)–(iv), we define an endotrivial S-module

$$M_S := \Omega^{n(E_1)}k \otimes N_2^{e_2} \otimes \dots \otimes N_r^{e_r},$$

where $e_i := (n(E_i) - n(E_1))/2p$. This has type function n, so we just need to check that M_S is G-stable. For every $g \in G$ and $V \leq {}^gS \cap S$, we have

$$\operatorname{res}^{{}^gS}_V(g \otimes_S M_S) \simeq \Omega^{n(V^g)}k \simeq \Omega^{n(V)}k \simeq \operatorname{res}^S_V M_S.$$

Either the subgroup ${}^gS \cap S$ is cyclic or $T({}^gS \cap S)$ is torsion-free. In either case, the injectivity of (8.6) shows that $g \otimes_S M_S$ and M_S are equivalent as ${}^gS \cap S$ modules. □

9. A local description of the group of endotrivial modules

We can provide another link between compatible actions and endotrivial modules by giving a description of $T(G)$ purely in terms of local information, *i.e.* in terms of N-stable elements of $T(P_0)$, where N is the normaliser in G of some chain of p-subgroups $P_0 < \dots < P_n$.

We first need to slightly expand the definition of Bredon cohomology that was given in Section 5 to cover more general coefficient systems.

9.1. **Definition.** Let ΔX denote the *category of simplices* of a G-space X, whose objects are simplices $\sigma : [n] \to X$ and whose morphisms $\sigma \to \sigma'$ are commutative diagrams

$$\begin{array}{ccc} [n] & \xrightarrow{\ \sigma\ } & \\ \downarrow & & X. \\ [n'] & \xrightarrow{\ \sigma'\ } & \end{array}$$

Let $(\Delta X)_G$ denote the Grothendieck construction for the left action of G on ΔX, whose objects are simplices $\sigma \in \Delta X$ and whose morphisms $\sigma \to \tau$ are pairs $(g \in G,\ f : g\sigma \to \tau)$, where f is a morphism in ΔX. A *(cohomological) G-local coefficient system* on X is a functor $F : (\Delta X)_G \to \mathrm{Ab}$.

Given a G-local coefficient system F on X, we define the non-equivariant cochains

$$\mathrm{C}^s(X; F) := \prod_{\sigma \in X_s} F(\sigma)$$

with differential given by $\delta(f)(\sigma) := \sum (-1)^i F(1, d^i)(f(d_i\sigma))$. We can define a G-action on $\mathrm{C}^s(X; F)$ where g acts by

$$F(g, \mathrm{id}_{g\sigma}) : F(\sigma) \to F(g\sigma)$$

on the component $F(\sigma)$. The *G-equivariant Bredon cohomology* of X with coefficients in F is then defined to be the cohomology of the cochain complex $\mathrm{C}^s_G(X; F) := \mathrm{C}^s(X; F)^G$.

We have a functor

$$\begin{aligned} (\Delta\mathscr{C})_G &\to \mathcal{O}(G)^{\mathrm{op}}, \\ \sigma &\mapsto G/G_\sigma \end{aligned}$$

where G_σ denotes the stabiliser of the simplex σ, and a morphism $(g, f : g\sigma \to \tau)$ is sent to the composite

$$G/G_\sigma \xleftarrow{g} G/G_{g\sigma} \xleftarrow{f} G/G_\tau.$$

In Section 5, we considered only *G-isotropy coefficient systems*, where $F : (\Delta\mathscr{C})_G \to \mathrm{Ab}$ is induced from a functor $\mathcal{O}(G)^{\mathrm{op}} \to \mathrm{Ab}$ by pulling back along the above functor $(\Delta\mathscr{C})_G \to \mathcal{O}(G)^{\mathrm{op}}$.

Given a functor $\bar{s}S_{\mathcal{S}} \to \mathrm{Ab}$, we can pull back along

$$\begin{aligned}(\Delta\mathscr{C})_G &\to \bar{s}S_{\mathscr{C}}\\ \sigma &\mapsto [\sigma]\end{aligned}$$

to obtain a G-local coefficient system. Bredon cohomology with G-isotropy coefficients is an invariant of the G-equivariant homotopy type of X, but this is no longer the case for a general G-local coefficient system.

9.2. **Lemma.** *Let C denote the functor $\bar{s}S_{\mathcal{S}}(G) \to \mathrm{Ab}$ that takes a chain $P_0 < \ldots < P_n$ to the group of isomorphism classes of endotrivial P_0-modules with a compatible $N_G(P_0 < \ldots < P_n)$-action. There is an isomorphism*

$$T(G) \xrightarrow{\cong} \lim_{\sigma \in \bar{s}S_{\mathcal{S}}(G)} C(\sigma)$$

induced by the restriction maps

$$\widetilde{\operatorname{res}}^{N_G(\sigma)}_{P_0} \operatorname{res}^G_{N_G(\sigma)} : T(G) \to C(\sigma).$$

Recall from Notation 6.14 that $\widetilde{\operatorname{res}}^N_P(M)$ denotes the module $\operatorname{res}^N_P(M)$ with a compatible N-action given by its forgotten N-module structure.

Proof. We introduce two other functors from $\bar{s}S_{\mathcal{S}}(G)$. We will use σ to refer to a chain of subgroups $P_0 < \ldots < P_n$. Firstly, let $H : \bar{s}S_{\mathcal{S}}(G) \to \mathrm{Ab}$ take σ to $\mathrm{H}^1(N_G(\sigma); k^\times)$. Secondly, let $T^{\mathrm{st}} : \bar{s}S_{\mathcal{S}}(G) \to \mathrm{Ab}$ take σ to the group of $N_G(\sigma)$-stable elements of $T(P_0)$.

We have a natural transformation $H \to C$ that takes a one-dimensional character φ to $\widetilde{\operatorname{res}}^{N(\sigma)}_{P_0}\varphi$. We also have a natural transformation $C \to T^{\mathrm{st}}$ that forgets the compatible action and remembers only the underlying endotrivial P_0-module.

These functors assemble into a short exact sequence of functors

$$0 \to H \to C \to T^{\mathrm{st}} \to 0.$$

Indeed, such a sequence is exact if and only if it is exact pointwise. Proposition 6.15(iii) implies that the first map is injective. The second map is surjective because we can equip any $N_G(\sigma)$-stable P_0-module M with a compatible action: by Lemma 6.5 it is enough to provide a compatible action on $\operatorname{res}^{P_0}_{Z(P_0)} M$, which is equivalent to $\operatorname{res}^G_{Z(P_0)} \Omega^n k$ for some n. Finally, exactness at the middle term follows because the kernel of the second map is precisely the group of compatible $N_G(\sigma)$-actions on k.

We therefore get an exact sequence

$$0 \to \lim_{\bar{s}S_{\mathcal{S}}(G)} H \to \lim_{\bar{s}S_{\mathcal{S}}(G)} C \to \lim_{\bar{s}S_{\mathcal{S}}(G)} T^{\mathrm{st}} \to \lim^1_{\bar{s}S_{\mathcal{S}}(G)} H,$$

which we compare to the exact sequence from Corollary 5.10:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(G,S) & \longrightarrow & T(G) & \longrightarrow & \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T & \xrightarrow{\beta} & \mathrm{H}^1_G(\mathcal{S}_p(G); \mathrm{H}^1(-; k^\times)) \\
 & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & \lim_{\bar{s}S_{\mathcal{S}}(G)} H & \longrightarrow & \lim_{\bar{s}S_{\mathcal{S}}(G)} C & \longrightarrow & \lim_{\bar{s}S_{\mathcal{S}}(G)} T^{\mathrm{st}} & \longrightarrow & \lim^1_{\bar{s}S_{\mathcal{S}}(G)} H.
\end{array}$$

The first vertical map is the isomorphism described by [Gro18, Theorem D]. It is induced by the maps sending a Sylow-trivial module M to the one-dimensional character given by $\hat{\mathrm{H}}^0(P_0; M)$, considered as a one-dimensional character of $N_G(\sigma)$. The second vertical map is induced by the composition $\widetilde{\operatorname{res}}^{N_G(\sigma)}_{P_0} \operatorname{res}^G_{N_G(\sigma)}$. The third vertical map is induced by projection onto $T(P_0)$, the image of which lands inside $T^{\mathrm{st}}(\sigma)$; its inverse is induced by

$$\begin{array}{ccc} \lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)} T^{\mathrm{st}}(-) & \dashrightarrow & \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \\ \downarrow & & \downarrow \\ T^{\mathrm{st}}([P]) & \longrightarrow & T(P). \end{array}$$

The fourth vertical map is the isomorphism described by [Gro02, Proposition 7.1], which specialises to the statement that

$$\lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)}{}^{*} F \cong \mathrm{H}^*_G(\mathcal{S}_p(G);\, F) \tag{9.3}$$

for any functor $F\colon \bar{s}\mathcal{S}_{\mathcal{S}}(G) \to \mathrm{Ab}$.

The rightmost square is the only one whose commutativity is not straightforward to check. Equation (9.3) implies that it is enough to check that the description of β given in Proposition 5.11 agrees with the boundary map arising from the short exact sequence of cochain complexes

$$0 \to \mathrm{C}^\bullet_G(\mathcal{S}_p(G);\, H) \to \mathrm{C}^\bullet_G(\mathcal{S}_p(G);\, C) \to \mathrm{C}^\bullet_G(\mathcal{S}_p(G);\, T^{\mathrm{st}}) \to 0.$$

We omit this check. The lemma then follows from the five lemma. □

The above lemma implies a similar statement for the functor $T(N_G(-))$:

9.4. **Theorem.** *There is an isomorphism*

$$T(G) \xrightarrow{\cong} \lim_{\sigma \in \bar{s}\mathcal{S}_{\mathcal{S}}(G)} T(N_G(\sigma))$$

induced by restriction.

Proof. We can factorise the map in Lemma 9.2 as

$$\begin{array}{ccccc} T(G) & & \xrightarrow{\quad\cong\quad} & & \lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)} C(\sigma) \\ & \searrow^{\operatorname{res}^G_{N_G(\sigma)}} & & \nearrow_{f} & \\ & & \lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)} T(N_G(\sigma)) & & \end{array}$$

so it is sufficient to show that the map f is injective. Suppose that $(M_\sigma) \in \lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)} T(N_G(\sigma))$ mapped to zero in $\lim_{\bar{s}\mathcal{S}_{\mathcal{S}}(G)} C(\sigma)$. We will start by showing that each M_σ is Sylow-trivial. Let $\sigma \in \bar{s}\mathcal{S}_{\mathcal{S}}(G)$, and let R denote a Sylow subgroup of $N_G(\sigma)$.

We claim that there is a zig-zag though simplices normalised by R that starts at σ and ends at a simplex in $\mathcal{S}_p(G)$ all of whose subgroups contain R. Let $\sigma = (P_0 < \ldots < P_n)$ and let i be maximal such that $R \not\leq P_i$. Since $R \leq N_G(\sigma)$, the group RP_i is still a p-subgroup of G with $R \leq N_G(RP_i)$. We can now replace P_i with RP_i by means of the following zig-zag in $\bar{s}\mathcal{S}_{\mathcal{S}}(G)$:

$$\begin{array}{ccc} \sigma = (\ldots < P_{i-1} < P_i < P_{i+1}R < \ldots) & & (\ldots < P_{i-1} < P_iR \leq P_{i+1}R < \ldots) =: \sigma' \\ \searrow & & \swarrow \\ & (\ldots < P_{i-1} < P_i < P_iR \leq P_{i+1}R < \ldots). & \end{array}$$

Note that σ' has strictly fewer vertices that do not contain R (and is of a smaller dimension if $P_iR = P_{i+1}R$). Since all of the simplices in the diagram are normalised by R, the modules M_σ and $M_{\sigma'}$ agree after restriction to R. By repeating this procedure, we find a simplex $\tau = (Q_0 <$

$\ldots < Q_m)$ with $R \leq Q_0$ such that M_σ and M_τ agree on restriction to R. By assumption, M_τ maps to zero in $C(\tau)$ and hence restricts trivially to R. Therefore M_σ is Sylow-trivial, as claimed.

Each $N_G(\sigma)$ has a non-trivial normal p-subgroup, so all of its Sylow-trivial modules are one-dimensional [MT07, Lemma 2.6]. Finally, by Proposition 6.15, if a one-dimensional character of $T(N_G(\sigma))$ maps to zero in $C(\sigma)$, then it must have been the trivial character. Therefore, f is injective as claimed. □

10. ENDOTRIVIAL MODULES FOR $\mathrm{PSL}_3(p)$

In this section, we apply our obstruction theory to the example of $G = \mathrm{PSL}_3(p)$, for $p \equiv 1 \bmod 3$, in which the obstruction to lifting G-stable endotrivial S-modules can be seen directly without reference to the cohomological obstruction classes. We will compute the obstruction group $\mathrm{H}^1_G(\mathcal{S}_p(G); \mathrm{H}^1(-; k^\times))$ and use orientations to determine exactly which G-stable endotrivial S-modules lift to $T(G)$.

We first collect some group-theoretic facts about the groups $\mathrm{SL}_3(p)$ and $\mathrm{PSL}_3(p)$ that we will need below. Recall that $O_p(H)$ denotes the p-core of H, *i.e.* its largest normal p-subgroup, and that a p-radical subgroup of G is a p-subgroup $P \leq G$ such that $P = O_p(N_G(P))$. We denote the collection of non-trivial p-radical subgroups of G by $\mathcal{B}_p(G)$.

10.1. **Lemma.** *Let* $\tilde{G} := \mathrm{SL}_3(p)$ *and* $G := \mathrm{PSL}_3(p)$, *for any prime* p.

(i) The non-trivial p*-radical subgroups of* G *and* $\tilde{G}$ *are (up to conjugacy) the Sylow subgroup* S*, consisting of the unipotent upper-triangular matrices, along with the rank 2 elementary abelian subgroups*

$$V := \begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{and} \quad W := \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}.$$

(ii) $N_{\tilde{G}}(V)$, $N_{\tilde{G}}(W)$ *and* $N_{\tilde{G}}(S)$ *are respectively given by the subgroups consisting of matrices with determinant one that are of the form*

$$\left(\begin{array}{c|cc} v & * & * \\ \hline & \multicolumn{2}{c}{A} \end{array}\right), \quad \left(\begin{array}{c|c} A & \begin{matrix} * \\ * \end{matrix} \\ \hline & w \end{array}\right), \quad \text{and} \quad \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}.$$

(iii) There are isomorphisms

$$\begin{array}{ccc} N_{\tilde{G}}(V)/V & \xrightarrow{\cong} & \mathrm{GL}_2(p) \\ \left(\begin{array}{c|cc} v & 0 & 0 \\ \hline & \multicolumn{2}{c}{A} \end{array}\right)V & \mapsto & A \end{array}$$

and

$$\begin{array}{ccc} N_{\tilde{G}}(W)/W & \xrightarrow{\cong} & \mathrm{GL}_2(p), \\ \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline & w \end{array}\right)W & \mapsto & A \end{array}$$

as well as an isomorphism

$$\begin{array}{ccc} N_{\tilde{G}}(S)/S & \xrightarrow{\cong} & \mathrm{GL}_1(p) \times \mathrm{GL}_1(p). \\ \begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}S & \mapsto & (v, w) \end{array}$$

(iv) *Suppose* $p \equiv 1 \bmod 3$*, and let* ω *be a primitive third root of unity in* $\mathbb{F}_p$*. The Weyl groups* $N_G(V)/V$ *and* $N_G(W)/W$ *both fit into a central* p'*-extension*

$$1 \to \mathrm{SL}_2(p) \to \ ? \ \to \mathrm{GL}_1(p)/\omega \to 1$$

induced by the isomorphisms of (iii). The quotient map is given by sending each of the specified coset representatives to $\det A$.

Proof.

(i) & (ii) Since we are working at the characteristic of $\tilde{G}$, we can apply a corollary of the Borel–Tits theorem [GLS98, Corollary 3.1.5] to deduce that if P is p-radical then $N_{\tilde{G}}(P)$ is a parabolic subgroup of $\tilde{G}$. The parabolic subgroups are classified up to conjugacy by [GLS98, Theorem 1.13.2]: we have a maximal torus given by the diagonal matrices and a Borel subgroup B given by the upper-triangular matrices, with respect to which the standard parabolics are $\tilde{G}$, B,

$$P_V := \left(\begin{array}{c|cc} v & * & * \\ \hline & \multicolumn{2}{c}{\mathrm{A}} \end{array}\right) \quad \text{and} \quad P_W := \left(\begin{array}{c|c} \mathrm{A} & \begin{matrix} * \\ * \end{matrix} \\ \hline & w \end{array}\right).$$

The corresponding p-radical subgroups are $1 = O_p(\tilde{G})$, $S = O_p(B)$, $V = O_p(P_V)$, and $W = O_p(P_W)$. This establishes the claim for $\tilde{G}$.

We now transfer the above results down to G. The kernel of the map $\pi : \tilde{G} \to G$ is a central p'-group, so $\pi^{-1}(P)$ has a unique Sylow p-subgroup (isomorphic to P) for any p-subgroup $P \leq G$. This implies that π induces a $\tilde{G}$-equivariant isomorphism of posets $\pi_* : \mathcal{S}_p(\tilde{G}) \to \mathcal{S}_p(G)$ and that $O_p(\pi(\tilde{H})) = \pi(O_p(\tilde{H}))$ for any $\tilde{H} \leq \tilde{G}$. It follows that π_* restricts to an isomorphism $\pi_* : \mathcal{B}_p(\tilde{G}) \to \mathcal{B}_p(G)$, completing the proof.

(iii) This is a straightforward calculation using (ii).

(iv) This follows from (iii) and the observation that the preimages of $\mathrm{SL}_2(p)$ in $N_{\tilde{G}}(V)$ and $N_{\tilde{G}}(W)$ intersect $\ker \pi$ trivially. □

Using the above computations, we give a necessary condition for a $\mathrm{PSL}_3(p)$-stable endotrivial S-module to lift to $T(\mathrm{PSL}_3(p))$. Afterwards, we use Proposition 7.5 to show that the condition is also sufficient.

10.2. *Example.* Let $p \equiv 1 \bmod 3$ and $G = \mathrm{PSL}_3(p)$. Let V, W and S be as in Lemma 10.1. Let $Z := Z(S) = V \cap W$, a cyclic subgroup of order p. We are interested in the image of the restriction map

$$T(G) \to \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-).$$

First consider $(M_P) \in \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-)$. By Dade's Theorem 2.8, we have $M_V \simeq \Omega^m k$ and $M_W \simeq \Omega^n k$ for some integers m and n. Since M_V and M_W are equivalent after restriction to Z and $T(Z) \cong \mathbb{Z}/2$, these integers satisfy $m \equiv n \bmod 2$.

Now instead consider $M \in T(G)$. We similarly have $\mathrm{res}^G_V M \simeq \Omega^m k$ and $\mathrm{res}^G_W M \simeq \Omega^n k$ for some integers m and n, but we will show that in this case $m \equiv n \bmod 6$. This difference arises from the non-trivial action of p'-elements in $N_G(Z)$. Since $p \equiv 1 \bmod 3$, there is a primitive third root of unity ω in $\mathbb{F}_p$. Let

$$x := \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$$

and note that x normalises Z, V and W.

We claim that $\mathrm{res}^G_{V\rtimes x} M \simeq \Omega^m k_{V\rtimes x}$, where the notation $V\rtimes x$ is shorthand for the semi-direct product $V \rtimes \langle x\rangle$. Indeed, we have a short exact sequence

$$0 \to T(V \rtimes x, V) \to T(V \rtimes x) \xrightarrow{\mathrm{res}} T(V) \to 0$$

which is split by $\Omega k_V \mapsto \Omega k_{V\rtimes x}$. This implies that $\Omega^{-m}\,\mathrm{res}^G_{V\rtimes x} M \in T(V\rtimes x, V)$. Since $V\rtimes x$ has a normal p-subgroup, $T(V \rtimes x, V)$ is isomorphic to the group of one-dimensional characters $\mathrm{H}^1(V \rtimes x;\, k^\times)$. Using the description of $T(V \rtimes x, V)$ found in [Gro18, Theorem A], we see $\Omega^{-m}\,\mathrm{res}^G_{V\rtimes x} M$ corresponds to the one-dimensional $(V\rtimes x)$-character given by Tate cohomology $\hat{\mathrm{H}}^0(V;\, \Omega^{-m}M) \cong \hat{\mathrm{H}}^m(V;\, M)$.

However, this $(V \rtimes x)$-character is the restriction of the $N_G(V)$-character $\hat{\mathrm{H}}^m(V;\, M)$, and by Lemma 10.1(iv) the image of x in $N_G(V)/V$ lies inside the subgroup $\mathrm{SL}_2(p)$. Since $\mathrm{SL}_2(p)$ is perfect, the $(V \rtimes x)$-character $\hat{\mathrm{H}}^m(V;\, M)$ is necessarily trivial, and $\mathrm{res}^G_{V\rtimes x} M \simeq \Omega^m k$ as claimed. The analogous statement holds for W and n.

We now consider the restriction of M to $Z \rtimes x$. We must have $\Omega^m k_{Z\rtimes x} \simeq \Omega^n k_{Z\rtimes x}$, *i.e.* m and n must be congruent modulo the periodicity of a $(Z \rtimes x)$-resolution of the trivial module. Identifying Z with the additive group of $\mathbb{F}_p$, the action of x on Z is given by multiplication by ω^2, so this periodicity is six and we have $m \equiv n \bmod 6$ as claimed.

Conversely, we now show that if $m \equiv n \bmod 6$, then we can lift $(M_P) \in \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-)$ to $T(G)$:

10.3. **Theorem.** *Let* $p \equiv 1 \pmod 3$ *and* $G = \mathrm{PSL}_3(p)$. *We have an exact sequence*

$$0 \to T(G) \to \lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-) \to \mathbb{Z}/3 \to 0.$$

Proof. Let V, W, S and ω be as in Lemma 10.1. In Example 10.2, we saw that every type function $n :\ \mathcal{S}_p(G) \to \mathbb{Z}$ of a G-module satisfies $n(V) \equiv n(W) \bmod 6$. We will show that this condition is also sufficient for lifting a G-stable endotrivial S-module to $T(G)$.

We apply Proposition 7.5 to $\mathcal{B}_p(G)$. We have already computed the p-radical subgroups of G up to G-conjugacy. The p-radical subgroup that has a cyclic centre is the Sylow, whose centre is

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

We have that $N_G(S)/S \cong (\mathrm{GL}_1(p) \times \mathrm{GL}_1(p))/\langle(\omega,\omega)\rangle$ with coset representatives

$$\begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}.$$

If we identify $Z(S)$ with $\mathbb{F}_p$, then right conjugation by the above coset representative induces multiplication by $v^{-1}w$. This determines the twisting character $\nu_S \in \mathrm{H}^1(N_G(S);\, k^\times)$.

Since $N_G(V < S) = N_G(S)$, to specify an orientation, it is sufficient to give characters $\varphi_V \in \mathrm{H}^1(N_G(V);\, k^\times)$ and $\varphi_W \in \mathrm{H}^1(N_G(W);\, k^\times)$ such that

$$\varphi_V = \nu_S^{(n(V)-n(W))/2}\varphi_W \tag{10.4}$$

in $\mathrm{H}^1(N_G(S);\, k^\times)$. We have $\mathrm{H}^1(N_G(V);\, k^\times) \cong \mathrm{Hom}(\mathrm{GL}_1(p)/\omega, k^\times)$, and we let ψ_V denote the image of φ_V under this isomorphism. Similarly, let ψ_W denote the image of φ_W under the analogous isomorphism for $N_G(W)$. When we restrict φ_V and φ_W to $N_G(S)$, we get

$$\varphi_V\left(\begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}\right) = \psi_V(v^{-1}) \quad \text{and} \quad \varphi_W\left(\begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}\right) = \psi_W(w^{-1}).$$

Write $n(V) - n(W) = 6\lambda$ for some $\lambda \in \mathbb{Z}$. We choose

$$\psi_V(x) := x^{3\lambda} \in k^\times \quad \text{and} \quad \psi_W(x) := x^{3\lambda} \in k^\times,$$

noting that both characters send ω to 1 as required. We have

$$\begin{aligned}(\nu_S^{3\lambda} \cdot \varphi_W)\left(\begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}\right) &= (v^{-1}w)^{3\lambda} \cdot w^{-3\lambda} \\ &= v^{-3\lambda} \\ &= \varphi_V\left(\begin{pmatrix} v & 0 & 0 \\ & (vw)^{-1} & 0 \\ & & w \end{pmatrix}\right),\end{aligned}$$

so Equation (10.4) is satisfied by these choices.

We have now shown that the congruence condition $n(V) \equiv n(W) \bmod 6$ is both necessary and sufficient for a G-stable endotrivial S-module to lift to $T(G)$, *i.e.* we have shown that the image of $T(G)$ has index three inside $\lim_{\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}}} T(-)$. Therefore the image of $\lim_{\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}}} T(-)$ inside the obstruction group $\mathrm{H}^1_G(\mathcal{B}_p(G); \mathrm{H}^1(-; k^\times))$ is of order three.

It remains to show that the obstruction group itself is of order three and to compute $T(G, S)$, which is isomorphic to $\mathrm{H}^0_G(\mathcal{B}_p(G); \mathrm{H}^1(-; k^\times))$ by [Gro18, §5]. This is a direct computation: the cochain complex $\mathrm{C}^\bullet(\mathcal{B}_p(G); \mathrm{H}^1(-; k^\times))$ is isomorphic to the complex

$$\begin{array}{ccc} \mathrm{H}^1(N_G(V); k^\times) & \searrow^{-\operatorname{res}^{N(V)}_{N(S)}} & \\ \oplus & & \\ \mathrm{H}^1(N_G(W); k^\times) & \xrightarrow[\operatorname{res}^{N(W)}_{N(S)}]{} & \mathrm{H}^1(N_G(S); k^\times) \\ \oplus & & \oplus \\ \mathrm{H}^1(N_G(S); k^\times) & \xrightarrow{\mathrm{id}} & \mathrm{H}^1(N_G(S); k^\times) \end{array}$$

and so splits off an acyclic summand (the bottom row). To compute $\mathrm{H}^0_G(\mathcal{B}_p(G); \mathrm{H}^1(-; k^\times))$ and $\mathrm{H}^1_G(\mathcal{B}_p(G); \mathrm{H}^1(-; k^\times))$ we therefore wish to determine the kernel and cokernel of the map to the top copy of $\mathrm{H}^1(N_G(S); k^\times)$.

We choose generators for $N_G(S)/S$:

$$\gamma_0 := \begin{pmatrix} \zeta & 0 & 0 \\ & \zeta^{-2} & 0 \\ & & \zeta \end{pmatrix} \quad \text{and} \quad \gamma_1 := \begin{pmatrix} \zeta & 0 & 0 \\ & \zeta^{-1} & 0 \\ & & 1 \end{pmatrix}$$

where ζ is a generator for $\mathbb{F}_p^\times$. The cohomology group $\mathrm{H}^1(N_G(S); k^\times)$ is then generated by the character γ_0^*, defined to send γ_0 to ζ^3 and γ_1 to 1, and the character γ_1^*, defined to send γ_0 to 1 and γ_1 to ζ.

By Lemma 10.1, the generator of $\mathrm{H}^1(N_G(W);\, k^\times)$ sends an element

$$\left(\begin{array}{c|c} \mathrm{A} & \begin{matrix}0\\0\end{matrix} \\ \hline & w \end{array}\right)$$

of $N_G(W)$ to $(\det A)^3$. This implies that the image of $\mathrm{res}^{N(W)}_{N(S)}$ is the subgroup generated by γ_0^*.

Similarly, the generator of $\mathrm{H}^1(N_G(V);\, k^\times)$ sends an element

$$\left(\begin{array}{c|c} v & 0 \quad 0 \\ \hline & \mathrm{A} \end{array}\right)$$

of $N_G(V)$ to $(\det A)^3$, so the image of $\mathrm{res}^{N(V)}_{N(S)}$ is the subgroup generated by $-\gamma_0^* - 3\gamma_1^*$.

Therefore, $\mathrm{H}^0_G(\mathcal{B}_p(G);\, \mathrm{H}^1(-;\, k^\times)) = 0$ and $\mathrm{H}^1_G(\mathcal{B}_p(G);\, \mathrm{H}^1(-;\, k^\times)) \cong \mathbb{Z}/3$, generated by γ_1^*. □

10.5. *Remark.* If $p \not\equiv 1 \pmod 3$, then $\mathrm{PSL}_3(p) = \mathrm{SL}_3(p)$ and an analogous calculation shows that $\mathrm{H}^1_G(\mathcal{B}_p(G);\, \mathrm{H}^1(-;\, k^\times)) = 0$, so the obstruction vanishes in this case. Similarly, if $n \leq 2$, then $\mathrm{PSL}_n(p)$ has a cyclic Sylow subgroup, while if $n \geq 4$, then $\mathrm{PSL}_n(p)$ has no rank two maximal elementary abelian subgroups. In either case, we can see directly that the restriction $T(G) \to T(S)$ is surjective. Therefore, the example of $\mathrm{PSL}_3(p)$ with $p \equiv 1 \pmod 3$ is the only case where we get interesting behaviour.

11. Torsion-free endotrivial modules

Let $TF(G)$ denote the torsion-free quotient of $T(G)$, *i.e.* the quotient by its torsion subgroup $TT(G)$. When p is odd and our obstructions vanish, we obtain an expression for the torsion-free endotrivial modules of G as a limit over the orbit category:

11.1. **Proposition.** *Let p be odd and suppose that there is a short exact sequence*

$$0 \to T(G,S) \to T(G) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \to 0,$$

as is for example the case when G has a non-trivial normal p-subgroup. Restriction induces an isomorphism

$$TF(G) \xrightarrow{\cong} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} TF(-).$$

Proof. Since $T(G,S)$ is finite [CMT14, Proposition 2.3], the short exact sequence given in the statement of the proposition implies that $TF(G)$ is isomorphic to the torsion-free quotient of $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$. We have a short exact sequence of functors from $\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}$

$$0 \to TT \to T \to TF \to 0$$

that induces an exact sequence

$$0 \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} TT(-) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \xrightarrow{f} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} TF(-) \to \lim^1_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} TT(-). \tag{11.2}$$

Since the first term of (11.2) is torsion and the third term is torsion-free, the image of the map f is isomorphic to the torsion-free quotient of $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$, which by the above argument is isomorphic to $TF(G)$. It therefore remains to show that f is surjective.

We have a commutative diagram

$$\begin{array}{ccc} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) & \hookrightarrow & T(S) \\ \downarrow f & & \downarrow \\ \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} TF(-) & \hookrightarrow & TF(S). \end{array}$$

Since the right-hand vertical map is surjective, let M_S be a lift to $T(S)$ of a G-stable element in $TF(S)$. It suffices to check that M_S is still G-stable. Let $g \in G$. Since the map

$$T(P) \to \prod_{V \in \mathcal{A}_p(P)} T(V)$$

is injective when P is cyclic (and p is odd) or when $T(P)$ is torsion free, it is enough to check that $g \otimes_S M_S$ and M_S restrict to the same element of $T(V)$ for every $V \in \mathcal{A}_p({}^gS \cap S)$.

If V is cyclic, then this follows because $\mathcal{A}_p(G)/G$ is contractible, and consequently either the restriction of M_S to every cyclic p-subgroup is equivalent to Ωk or its restriction to every cyclic p-subgroup is equivalent to k.

Otherwise, $TF(V) = T(V)$ and so

$$\mathrm{res}^{{}^gS}_V (g \otimes_S M_S) \simeq \mathrm{res}^S_V M_S,$$

because M_S is a G-stable element of $TF(S)$. □

In [CMT14], Carlson–Mazza–Thévenaz asked several questions about the behaviour of the torsion-free quotient of $T(G)$. We are able to answer these questions at certain primes using Theorem 10.3, which computed the obstruction group for $\mathrm{PSL}_3(p)$. A homomorphism $\phi : G \to G'$ is said to preserve p-fusion if and only if it restricts to an isomorphism between some Sylow p-subgroups $S \le G$ and $S' \le G'$ and induces an equivalence $\mathcal{F}_{\mathcal{S}}(G) \to \mathcal{F}_{\mathcal{S}}(G')$ of fusion categories.

In [CMT14, Conjecture 10.1], Carlson–Mazza–Thévenaz asked whether ϕ controlling fusion implies that $\phi^* : TF(G') \to TF(G)$ is an isomorphism. The following theorem shows that this is true for $p = 2$ but not in general:

11.3. **Theorem.** *Suppose that $\phi : G \to G'$ is a group homomorphism that controls p-fusion. The induced homomorphism $\phi^* : TF(G') \to TF(G)$ is an isomorphism if $p = 2$. If p is odd, this is not necessarily the case; for example, when $p \equiv 1 \bmod 3$ the quotient map $\phi : \mathrm{SL}_3(p) \to \mathrm{PSL}_3(p)$ preserves fusion but the induced map $\phi^* : TF(\mathrm{PSL}_3(p)) \hookrightarrow TF(\mathrm{SL}_3(p))$ is the inclusion of a subgroup of index three.*

Proof. When $p = 2$, we have a short exact sequence

$$0 \to T(G, S) \to T(G) \to \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \to 0$$

by Corollary 7.12. Since $T(G, S)$ is finite [CMT14, Proposition 2.3], we have that $TF(G)$ is isomorphic to the torsion-free part of $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$. There is a zig-zag of functors $\mathcal{O}_{\mathcal{S}}(G) \leftarrow \mathcal{T}_{\mathcal{S}}(G) \to \mathcal{F}_{\mathcal{S}}(G)$, which induces an isomorphism $\lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-) \cong \lim_{\mathcal{F}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-)$. Since ϕ preserves p-fusion, it induces an equivalence on fusion categories, so we deduce that

$$\lim_{\mathcal{O}_{\mathcal{S}}(G')^{\mathrm{op}}} T(-) \cong \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} T(-).$$

Therefore $\phi^* : TF(G') \to TF(G)$ is an isomorphism, completing the proof of the case where $p = 2$.

Now let $\tilde{G} = \mathrm{SL}_3(p)$ and $G = \mathrm{PSL}_3(p)$ for $p \equiv 1 \bmod 3$. One can directly calculate that the obstruction group $\mathrm{H}^1_{\tilde{G}}(\mathcal{B}_p(\tilde{G}); \mathrm{H}^1(-; k^\times))$ vanishes and that $T(\tilde{G}, S) = 0$ by using the method that was used in Theorem 10.3 to carry out the analogous computation for G; the normalisers that appear were computed in Lemma 10.1.

We obtain a diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(G) & \longrightarrow & \lim_{\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}}} T(-) & \longrightarrow & \mathbb{Z}/3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\ 0 & \longrightarrow & T(\tilde{G}) & \xrightarrow{\cong} & \lim_{\mathcal{O}_\mathcal{S}(\tilde{G})^{\mathrm{op}}} T(-) & \longrightarrow & 0 & & \end{array}$$

The middle vertical map is an isomorphism because ϕ preserves p-fusion. We see that $\mathrm{Im}(\phi^*)$ is a subgroup of index three. □

In [CMT14, Conjecture 9.2], Carlson–Mazza–Thévenaz asked whether there is a torsion-free subgroup $F \leq T(G)$ such that F consists of modules lying in the principal block of G and $T(G) = TT(G) \oplus F$. The following theorem shows that this is not true when $G = \mathrm{SL}_3(p)$ and $p \equiv 1 \pmod 3$:

11.4. **Theorem.** *When $p \equiv 1 \bmod 3$, the torsion subgroup $TT(\mathrm{SL}_3(p))$ is zero, but the subgroup of $T(G)$ consisting of modules lying in the principal block has index at least three.*

Proof. We again let $\tilde{G} = \mathrm{SL}_3(p)$, $G = \mathrm{PSL}_3(p)$, and ϕ be the quotient map $\mathrm{SL}_3(p) \to \mathrm{PSL}_3(p)$. For any finite group H, the p'-core $O_{p'}(H)$ acts trivially on kH-modules lying in the principal block [HB82, Lemma 13.1], and here $O_{p'}(\tilde{G}) = Z(\tilde{G})$. The kernel of ϕ is also equal to $Z(\tilde{G})$, being the group generated by

$$\begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega \end{pmatrix}$$

for ω a primitive third root of unity in $\mathbb{F}_p$. Therefore, if $O_{p'}(\tilde{G})$ acts trivially on a $\tilde{G}$-module, then that module is inflated from G. Since not every module in $T(\tilde{G})$ is inflated from G, there must be endotrivial $\tilde{G}$-modules that do not lie in the principal block. □

11.5. *Remark.* Theorem 4.5 shows that a weaker form of the conjecture relating to fusion systems does hold. If instead we ask that ϕ induce an equivalence on orbit categories $\mathcal{O}_\mathcal{S}(G) \to \mathcal{O}_\mathcal{S}(G')$, then the five lemma applied to the exact sequence

$$0 \to \mathrm{H}^1(\mathcal{O}_\mathcal{S}(G); k^\times) \to T(G) \to \lim_{\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}}} T(-) \to \mathrm{H}^2(\mathcal{O}_\mathcal{S}(G); k^\times)$$

shows that ϕ induces an isomorphism on $T(G)$.

11.6. *Remark.* The fact that all G-stable endotrivial S-modules lift to $T(G)$ when $p = 2$ does not seem to say anything positive about the conjecture relating to principal blocks. Since p-groups only have one block, the information about the block of an indecomposable kG-module is somehow contained in the coherence data of the corresponding object of $\mathrm{holim}_{\mathcal{O}_\mathcal{S}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP}$, not its restrictions to StMod_{kP}.

12. Endotrivial modules for the alternating and symmetric groups

Here we use orientations in order to compute the image of $T(G)$ in $T(S)$ where G is either the symmetric group Σ_n or the alternating group A_n, the degree n satisfies $p^2 \leq n < p^2 + p$, and p is odd. Note that if $n < p^2$, then $TF(G) = 0$, while when $p^2 + p \leq n$, we have $TF(G) \cong \mathbb{Z}$ generated by Ωk. The case $p^2 \leq n < p^2 + p$ is considered in [CHM10], where they calculate that $TF(G) \cong \mathbb{Z}^2$ and provide bounds on the type of the generators. We provide precise values for the type of the generators in Theorem 12.1. The software package [GAP] was invaluable for working out the details of the computations below.

When $p^2 \leq n < p^2 + p$ and p is odd, both Σ_n and A_n have two maximal elementary abelian subgroups (up to conjugacy), which we denote E_1 and E_2. We describe these subgroups after the statement of the theorem.

12.1. **Theorem.** *Let G denote either the symmetric group Σ_n or the alternating group A_n, where $p^2 \leq n < p^2 + p$ and p is odd. We have $TF(G) \cong \mathbb{Z}^2$, generated by $[\Omega k]$ and $[M]$ for some module M satisfying*

$$\operatorname{res}^G_{E_1} M \simeq k \quad \textit{and} \quad \operatorname{res}^G_{E_2} M \simeq \Omega^{2p(p-1)} k.$$

Theorem 6.1 in [CHM10] proves that

$$\operatorname{res}^G_{E_1} M \simeq k \quad \text{and} \quad \operatorname{res}^G_{E_2} M \simeq \Omega^{2pr} k$$

for some r dividing $p - 1$. Our contribution is to show that $r = p - 1$. All of the work in the proof of the theorem is for the case $G = A_n$, which we deal with in Proposition 12.10; the case $G = \Sigma_n$ follows as a corollary. To prove the proposition, we require some understanding of the local structure of Σ_n and A_n, which we discuss first (following the description in [CHM10, §6]).

12.2. **Assumption.** Without further comment, we assume throughout the remainder of this section that $p^2 \leq n < p^2 + p$ and that p is odd.

Define elements in Σ_n for $i = 1, \dots, p$ by

$$x_i = (ip - p + 1, \dots, ip) \quad \text{and} \quad y = (1, p + 1, \dots, p^2 - p + 1) \dots (p, 2p, \dots, p^2),$$

where we consider Σ_n as acting on $\{1, \dots, n\}$ on the right. These $p + 1$ elements generate the Sylow subgroup $S \cong C_p \wr C_p$ of both Σ_n and A_n. Let E_1 be the base subgroup of the wreath product, *i.e.* $E_1 \coloneqq \langle x_1, \dots, x_p \rangle$, and let $E_2 \coloneqq \langle x_1 \dots x_p, y \rangle$. The subgroups E_1 and E_2 are the maximal elementary abelian subgroups of S up to conjugacy, and E_1 is clearly a maximal elementary abelian subgroup of A_n (and hence Σ_n). E_2 cannot be subconjugate to E_1, since all non-identity elements of E_2 are a product of p p-cycles and E_1 has a subgroup $E_1 \cap \Sigma_{p^2-p}$ that is of index p and has no such elements. Therefore E_2 is also a maximal elementary abelian subgroup of A_n (and hence Σ_n).

We will also use the following notation: there is a subgroup $\Sigma_p \wr \Sigma_p \leq \Sigma_n$, where the i^{th} base factor is viewed as acting on the set $\{ip - p + 1, \dots, ip\}$. We let $\beta_i : \Sigma_p \to \Sigma_p \wr \Sigma_p$ denote the inclusion of the i^{th} base factor and $\tau : \Sigma_p \to \Sigma_p \wr \Sigma_p$ denote the inclusion of the twisting factor. With this notation, $x_i = \beta_i((1, \dots, p))$ and $y = \tau((1, \dots, p))$. Note that $\operatorname{sgn} \tau(\sigma) = \operatorname{sgn} \beta(\sigma) = \operatorname{sgn} \sigma$.

12.3. *Example.* In the smallest possible case, we have $p = 3$ and $n = 9$. The elements defined above are $x_1 = (123)$, $x_2 = (456)$, $x_3 = (789)$, and $y = (147)(258)(369)$. The action of y cyclically permutes the x_i.

We will work with the collection $\mathcal{B}_p(G)$ of p-radical subgroups, so start by determining this collection up to conjugacy:

12.4. **Lemma.** *Up to G-conjugacy, the p-radical subgroups of $G = \Sigma_n$ and $G = A_n$ are:*

(i) the Sylow subgroup S,

(ii) the rank two elementary abelian subgroup E_2, and

(iii) the rank i elementary abelian subgroup $R_s := \langle x_1, \ldots, x_s \rangle$ for $1 \le i \le p$,

except when $p = 3$ and $n = 9$, in which case R_2 is not p-radical.

12.5. *Remark.* Note that $R_p = E_1$.

Proof of Lemma 12.4. We use the classification of p-radical subgroups of Σ_n found in [AF90, (2A)], which we now summarise: for $c \ge 1$, let B_c denote the (unique up to conjugacy) transitive subgroup of Σ_{p^c} that is isomorphic to the elementary abelian group of order p^c. This is given by the permutation representation arising from the right regular action of the elementary abelian group on itself. Note that B_1 is conjugate to $\langle x_1 \rangle$ and B_2 is conjugate to E_2.

For a tuple of positive integers $(c_1, \ldots, c_t)$, let $B_{(c_1,\ldots,c_t)}$ denote the iterated wreath product $B_{c_1} \wr \ldots \wr B_{c_t}$. This group embeds uniquely up to conjugacy as a transitive subgroup of Σ_{p^d}, where $d = c_1 + \ldots + c_t$. The subgroup $B_{(c_1,\ldots,c_t)} \le \Sigma_{p^d}$ is said to be a *basic* subgroup, whose *degree* is p^d and whose *length* is t.

Any p-radical subgroup R of $\mathrm{Sym}(V)$ splits as

$$R \cong T_0 \times \ldots \times T_s \tag{12.6}$$

corresponding to a partition

$$V = V_0 \amalg \ldots \amalg V_s.$$

T_0 is the trivial subgroup of $\mathrm{Sym}(V_0)$, and each T_i for $i > 0$ is a basic subgroup of $\mathrm{Sym}(V_i)$.

We note that $B_{(1,1,1)}$ has order $(p^p \cdot p)^p \cdot p = p^{p^2+p+1}$, which is strictly greater than $|S| = p^{p+1}$, so any basic subgroup in a decomposition of a p-radical subgroup of Σ_n has length at most 2. The basic subgroups we need to consider are therefore

(i) $B_{(1,1)} \cong C_p \wr C_p \cong S$ of degree p^2,

(ii) $B_{(1)} \cong C_p \cong \langle x_1 \rangle$ of degree p, and

(iii) $B_{(2)} \cong C_{p^2} \cong E_2$ of degree p^2.

Σ_n itself has degree strictly less than $p^2 + p$, so the possible candidates for p-radical subgroups of Σ_n are $B_{(1,1)}$, $B_{(2)}$, and a direct product of copies of $B_{(1)}$. It is not hard to deduce directly that the p-radical subgroups of Σ_n are as claimed in the statement of the lemma; however, any p-radical P in A_n is a p-radical in Σ_n, since

$$P \le A_n \cap O_p(N_{\Sigma_n}(P)) \le O_p(N_{A_n}(P)) = P$$

and a p-group has no subgroups of index two. Therefore, it is sufficient to check that each of our candidates is radical in A_n and that none of their Σ_n-conjugacy classes splits in two in A_n.

We first show the statement about conjugacy classes. For the Sylow subgroup there is nothing to check. By the orbit-stabiliser theorem it is enough to show that the stabiliser in Σ_n of each other p-radical subgroup P is strictly larger than its stabiliser in A_n. That is, we seek an element of Σ_n that normalises P and does not lie in A_n. Let $\xi \in N_{\Sigma_p}(C_p)$ represent a generator of the quotient $N_{\Sigma_p}(C_p)/C_p$. Note that ξ is an odd permutation, since the conjugacy class $C_p \setminus \{1\}$ splits into two conjugacy classes in A_p. The element $\beta_1(\xi) \ldots \beta_p(\xi)$ normalises $\langle y \rangle$, $\langle x_1 \ldots x_p \rangle$, and each $\langle x_i \rangle$. Moreover, it does not lie in A_n. Therefore, the Σ_n-conjugacy class of each of the subgroups listed above is the same as its A_n-conjugacy class.

It remains to check which subgroups in the above list are radical in A_n. For the Sylow subgroup there is nothing to check. For R_s we claim that

$$N_{\Sigma(n)}(R_s) \cong (N_{\Sigma(p)}(C_p) \wr \Sigma_s) \times \Sigma_{n-ps}. \tag{12.7}$$

Indeed, since conjugation preserves cycle type, any $\sigma \in \Sigma_{ps}$ that normalises R_s must satisfy ${x_i}^{\sigma} = {x_{\eta(i)}}^{n_i}$ for some permutation $\eta \in \Sigma_s$. The element $\sigma \cdot \tau(\eta)^{-1}$ normalises each $\langle x_i \rangle$, proving the claim. Write $W_G(H) := N_G(H)/H$ for the Weyl group of H in G. We deduce from Equation (12.7) that

$$W_{\Sigma(n)}(R_s) \cong (\mathbb{F}_p^\times \wr \Sigma_s) \times \Sigma_{n-ps},$$

and we note that $W_{A_n}(R_s)$ is an index two subgroup of this. The order of $W_{\Sigma_n}(R_s)$ is not divisible by p^2, so if $O_p(W_{A_n}(R_s)$ were non-trivial then it would be a Sylow subgroup of order p; in particular, $W_{A_n}(R_s)$ would have a unique Sylow p-subgroup, and hence so would $W_{\Sigma_n}(R_s)$. Therefore, to show that R_s is p-radical in A_n, it is enough to show $W_{\Sigma_n}(R_s)$ does not have a unique subgroup of order p. We prove this case-by-case:

- (i) If $p > 3$ and $s = p$, then there are $(p-2)!$ subgroups of order p in the Σ_s of the wreath product.
- (ii) If $p > 3$ and $s < p$, then the Σ_{n-ps} factor of the direct product contains a copy of Σ_p.
- (iii) If $p = 3$ and $s = p$, then $\langle y \rangle$ and $\langle \beta_1(\xi)^{-1}\beta_2(\xi)y \rangle$ are different subgroups of order p in $W_{\Sigma_n}(R_3)$.
- (iv) If $p = 3$, $s < p$, and $n > 3s + 3$, then the Σ_{n-ps} factor of the direct product contains a copy of Σ_4.

(The only case not covered is when $p = 3$, $s = 2$, and $n = 9$.) It follows that in all of the above cases, $O_p(W_{A(n)}(R_s)) = 1$ and R_s is p-radical in A_n. In the exceptional case, $W_{A(9)}(R_2) \cong D_{24}$ and $W_{\Sigma(9)}(R_2) \cong D_8 \times \Sigma_3$, both of which have a normal Sylow 3-subgroup.

Finally, we check that E_2 is p-radical in A_n. By [AF90, (2.1)], the Weyl group of $B_{(c_1,\ldots,c_t)}$ in Σ_{p^d} is given by

$$W_{\Sigma(p^d)}(B_{(c_1,\ldots,c_t)}) \cong \mathrm{GL}_{c_1}(p) \times \mathrm{GL}_{c_2}(p) \times \ldots \times \mathrm{GL}_{c_t}(p). \tag{12.8}$$

Since $E_2 = B_{(2)}$, we have

$$W_{\Sigma(n)}(E_2) \cong \mathrm{GL}_2(p) \times \Sigma_{n-p^2}.$$

The Σ_{n-p^2} factor is a p'-group, and $O_p(\mathrm{GL}_2(p)) = 1$ because the upper-triangular unitary matrices and lower-triangular unitary matrices are two Sylow subgroups whose intersection is trivial. ☐

For any group G, let $G_{p'}$ denote the quotient of G by the smallest normal subgroup $O^{p'}(G)$ such that $G/O^{p'}(G)$ is a p'-group; equivalently, $G_{p'}$ is the quotient of G by the normal subgroup generated by all elements of p-power order.

12.9. **Lemma.** *Let $p \geq 5$. The group $\mathrm{H}_1(N_{A(p^2)}(E_2))_{p'}$ is isomorphic to a subgroup of $\mathbb{F}_p^\times$. With this identification the canonical map*

$$N_{A(p^2)}(E_2) \to \mathrm{H}_1(N_{A(p^2)}(E_2))_{p'} \leq \mathbb{F}_p^\times$$

takes an element to the determinant of the matrix that represents it with respect to the basis $\{x_1 \ldots x_p, y\}$ of E_2, and the image of the canonical map is the subgroup of squares in $\mathbb{F}_p^\times$.

Proof. We have $W_{\Sigma(p^2)}(E_2) \cong \mathrm{GL}_2(p)$ by Equation (12.8). An element of $N_{\Sigma(p^2)}(E_2)$ is sent to the matrix representing it with respect to the basis $\{x_1 \ldots x_p, y\}$. The functors $\mathrm{H}_1(-)$ and $(-)_{p'}$ commute with each other (up to natural isomorphism), so $\mathrm{H}_1(N_{A(p^2)}(E_2))_{p'} \cong \mathrm{H}_1(W_{A(p^2)}(E_2))$ via an isomorphism that commutes with the quotient maps from $N_{A(p^2)}(E_2)$. The Weyl group

$W_{A(p^2)}(E_2)$ identifies with a subgroup $W \leq \mathrm{GL}_2(p)$ of index two, and it is enough to show that the derived subgroup of W is $\mathrm{SL}_2(p)$

We will first show that $\mathrm{SL}_2(p) \leq W$. We have $|\mathrm{SL}_2(p) : W \cap \mathrm{SL}_2(p)| \leq 2$, so $1 \neq W \cap \mathrm{SL}_2(p) \trianglelefteq \mathrm{SL}_2(p)$. We have a subnormal series of $\mathrm{SL}_2(p)$ given by

$$1 \underset{C_2}{\trianglelefteq} \{\pm I\} \underset{\mathrm{PSL}_2(p)}{\trianglelefteq} \mathrm{SL}_2(p).$$

We see that $\{\pm I\} \leq W$, so $(W \cap \mathrm{SL}_2(p))/\{\pm I\} \leq \mathrm{PSL}_2(p)$ is a subgroup of index at most two. However, $\mathrm{PSL}_2(p)$ is simple (since $p \geq 5$) so has no index two subgroups. Therefore, $W \cap \mathrm{SL}_2(p) = \mathrm{SL}_2(p)$, *i.e.* $\mathrm{SL}_2(p) \leq W$.

We deduce that $W/\mathrm{SL}_2(p) \leq \mathbb{F}_p^\times$ is an index two subgroup, and hence that W is the preimage (under the determinant homomorphism) of the set of squares in $\mathbb{F}_p^\times$. The group $\mathrm{SL}_2(p)$ is perfect, since $p \geq 5$, so is contained in the derived subgroup of W. Therefore, it is equal to the derived subgroup, and we see that $\mathrm{H}_1(W_{A(p^2)}(E_2))$ is the set of squares in $\mathbb{F}_p^\times$. □

12.10. **Proposition.** *If $M \in T(A_n)$ is of type $(0, 2pr)$, then $p-1$ divides r.*

Proof. Write $G := A_n$ for brevity. Note that $\mathcal{B}_p(G)$ satisfies the condition on $\mathscr{C}$ found in Proposition 7.5: for a 1-simplex $P < Q$ in $\mathcal{B}_p(G)$, if Q is elementary abelian then we can set $V := Q$. Otherwise, by conjugating $P < Q$ we can assume that $Q = S$. If P is conjugate to E_2, then we can set $V := P$. The remaining case is where P is a p-radical subgroup contained in S and subconjugate to E_1, and by comparing cycle types we see it must actually be a subgroup of E_1. In this case, we claim that we can take $V := E_1$. It is clear that E_1 contains both $Z(P) = P$ and $Z(S) = E_1 \cap E_2$, so we just need to check that $N_G(P < S)$ normalises E_1. In fact, the normaliser of S is contained in the normaliser of E_1, because E_1 is generated by the elements of S that are p-cycles. Therefore, $\mathcal{B}_p(G)$ satisfies the condition in Proposition 7.5.

Let $n : \mathcal{S}_p(G) \to \mathbb{Z}$ be a type function for M with $n(R_s) = n(E_1) = n(S) = 0$ and $n(E_2) = 2pr$. The strategy of the proof will be to compare the characters of an orientation on four p-radical subgroups: R_{p-2}, E_1, S, and E_2. We will show that it is impossible to specify an orientation for n unless $p-1$ divides r. Note that when $p = 3$ and $n = 9$, the fact that R_2 is not p-radical does not cause any issues, since we only use R_1 and R_3 (which is equal to E_1).

Let $\xi \in N_{\Sigma_p}(C_p)$ represent a generator of the quotient $N_{\Sigma_p}(C_p)/C_p$, and let $\lambda \in \mathbb{F}_p^\times$ be the generator corresponding to it under the canonical isomorphism $N_{\Sigma_p}(C_p)/C_p \cong \mathbb{F}_p^\times$ given by right conjugation. When $p = 3$, we will specifically choose $\xi = (2,3)$; this choice avoids problems later in the proof due to A_3 not being perfect. As argued for previously, ξ is an odd permutation. We give names to some elements of G:

$$a := \left(\prod_{1 \leq i \leq p} \beta_i(\xi)\right) \cdot \tau(\xi), \qquad b := \prod_{1 \leq i \leq p} \beta_i(\xi)^2,$$
$$a' := \beta_p(\xi) \cdot \tau((p-1, p)), \text{ and } b' := \beta_p(\xi)^2.$$

An overview of the proof is given in Figure 1. All boxes represent an element of $\mathbb{F}_p^\times$, and arrows indicate equality; the arrow is labelled by the compatibility condition that we are using in that step. The argument depends on several computations in homology, which we postpone until we have described the outline. We start at the top of the diagram, with the observation that $[a] = [b]$ in $\mathrm{H}_1(N_G(E_2))_{p'}$, where as before the subscript p' denotes the p'-quotient. This implies that the value the character φ_{E_2} takes on the elements a and b must agree. Next we use the condition from Proposition 7.5 that

$$\varphi_{E_2} = \nu_S^{pr} \varphi_S$$

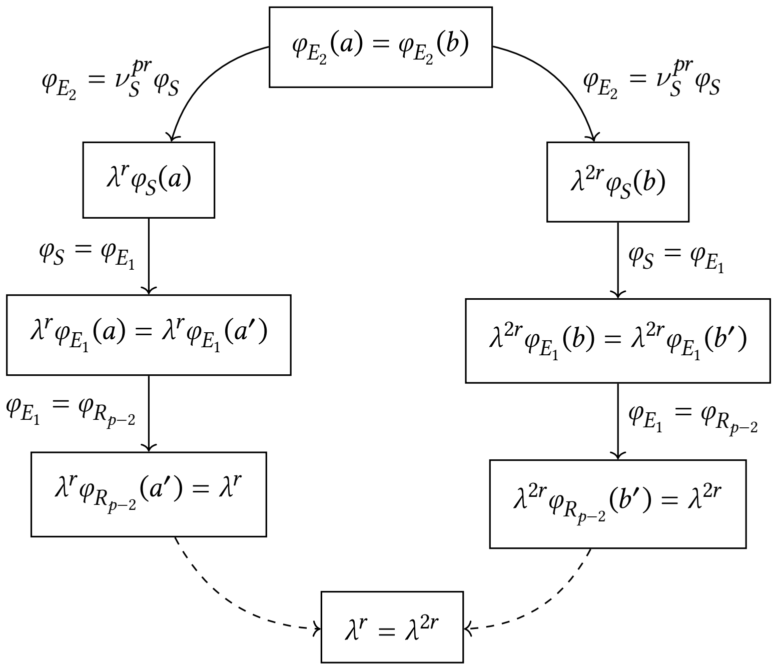


FIGURE 1. Overview of the comparison of characters for the alternating group.

as characters of $N_G(E_2 < S)$, but crucially the twisting character ν_S takes different values on a and b: the homology classes $[a]$ and $[b]$ do *not* agree in $\mathrm{H}_1(N_G(S))_{p'}$. We have $\nu_S(a) = \lambda$ and $\nu_S(b) = \lambda^2$. We now use the compatibility condition for orientation characters again: $\varphi_S = \varphi_{E_1}$ as characters of $N_G(E_1 < S)$, so $\varphi_S(a) = \varphi_{E_1}(a)$ and similarly for b. In $\mathrm{H}_1(N_G(E_1)_{p'}$ we have $[a] = [a']$ and $[b] = [b']$, so $\varphi_{E_1}(a) = \varphi_{E_1}(a')$ and similarly for b and b'. Finally, we use the compatibility condition one last time to pass to R_{p-2}. Here, however, we have $[a'] = [b'] = 0$ in $\mathrm{H}_1(N_G(R_{p-2}))$. We have therefore shown that $\lambda^r = \lambda^{2r}$; since λ is a generator for $\mathbb{F}_p^\times$, this can only be the case if $p - 1$ divides r.

It remains to justify all the claims made in the above argument. We start with an implicit claim, namely that $a, b \in N_G(E_1) \cap N_G(S) \cap N_G(E_2)$ and $a', b' \in N_G(R_{p-2}) \cap N_G(E_1)$. This is a straightforward check, since we know how $\beta_i(\xi)$, $\tau(\xi)$, and $\tau((p-1, p))$ act on the elements x_i and y.

Secondly, we show that $[a] = [b]$ in $\mathrm{H}_1(N_G(E_2))_{p'}$. When $p \geq 5$, we use Lemma 12.9: a and b represent the same class in $\mathrm{H}_1(N_{A(p^2)}(E_2))_{p'}$ if they correspond to matrices with the same determinant. With respect to the basis $\{x_1 \dots x_p, y\}$, the elements a and b act by the matrices

$$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda^2 & \\ & 1 \end{pmatrix} \tag{12.11}$$

respectively. When $p = 3$, it can be checked that $b = 1$ and

$$a = (2,3)(4,7)(5,9)(6,8) = [(2,4,3,7)(5,6,9,8),\ (2,5,3,9)(4,8,7,6)]$$

where both of the elements appearing in the commutator normalise E_2. The matrices in (12.11) also tell us the value of the twisting character ν_S on the elements a and b. The twisting character is determined by the right action of $N_G(S)$ on $Z(S) = \langle x_1 \dots x_p\rangle$, so its value is given by the top-left corner of the matrix. We have $\nu_S(a) = \lambda$ and $\nu_S(b) = \lambda^2$.

Thirdly, we show that $[a'] = [a]$ and $[b'] = [b]$ in $\mathrm{H}_1(N_G(E_1))_{p'}$. We have

$$\beta_i(\xi)^{-1} \cdot \beta_j(\xi) = [\beta_p(\xi)\beta_i(\xi),\ \ \beta_p(\xi)\tau((i,j))] \in [N_G(E_1), N_G(E_1)]$$

for any distinct i and j with $i, j < p$, so $[\beta_i(\xi)] = [\beta_j(\xi)]$ in $\mathrm{H}_1(N_G(E_1))_{p'}$. Since $\xi^{p-1} \in C_p$, this implies that $[b'] = [b]$ and $[a'^{-1}a] = [\tau((p-1,p)\cdot\xi)]$ once we have passed to the p'-quotient. When $p = 3$, we chose $\xi = (2,3)$, while when $p \geq 5$ we have $\tau(A_p) = [\tau(A_p), \tau(A_p)] \leq [N_G(E_1), N_G(E_1)]$. In either case, $[a'] = [a]$ as claimed.

Finally, we observe that $[a'] = [b'] = 0$ in $\mathrm{H}_1(N_G(R_{p-2}))$. Indeed, both a' and b' are contained in $\mathrm{Alt}(\{p^2 - 2p + 1, \dots, p^2\}) \cong A_{2p}$, which is a perfect subgroup of $N_G(R_{p-2})$. □

Proof of Theorem 12.1. As noted in the introduction to this section, all that is needed in addition to [CHM10, Theorem 6.1] is to rule out the existence of modules of type $(0, 2pr)$ for $0 < r < p-1$. Proposition 12.10 does this for the alternating group, and restricting from $T(\Sigma_n)$ to $T(A_n)$ preserves the type of a module so the result follows for the symmetric group too. □

Appendix A. Comparison with an obstruction due to Balmer

Let $H \leq G$ be a subgroup whose index in G is prime to p. In Theorem 10.7 of [Bal15], Balmer provides an abstract obstruction for lifting a kH-module to a kG-module, which lives in the second Čech cohomology group of a certain cover $\mathcal{U}$ in his "sipp" topology on finite G-sets. The goal of this section is to relate his obstruction to our class $\alpha \in \mathrm{H}^2(\mathcal{O}_\mathcal{S}(G); k^\times)$. We recall the setup of [Bal15] and show that the Čech cohomology groups $\check{\mathrm{H}}^*(\mathcal{U}; \mathbb{G}_m)$ are isomorphic to the cohomology groups $\mathrm{H}^*(\mathcal{O}_\mathcal{S}(G); k^\times)$ in the case where $H = S$. We also show that our obstruction class α_M is sent to the obstruction of [Bal15, Theorem 10.7] under this isomorphism.

Let $\mathcal{C}$ be a category, let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be a presheaf of abelian groups, and let $U \to X$ be a morphism for which the iterated pullbacks $U \times_X \dots \times_X U$ exist in $\mathcal{C}$; for example, we can take $U \to X$ to be a cover in a Grothendieck topology on $\mathcal{C}$. We get a simplicial diagram

$$\cdots \rightrightarrows\!\!\!\!\rightrightarrows U \times_X U \times_X U \;\substack{\to\\\to\\\to}\; U \times_X U \rightrightarrows U$$

in $\mathcal{C}$, with the face maps induced by projections and degeneracies induced by the diagonal map. Applying F to the above diagram gives a cosimplicial abelian group and hence a cochain complex (whose differential is given by the alternating sum of face maps). We define the *Čech cohomology* of $\mathcal{U} := \{U \to X\}$ with coefficients in F, written $\check{\mathrm{H}}^*(\mathcal{U}; F)$, to be the cohomology of this cochain complex.

The *sipp-topology* is the Grothendieck topology on the category G-Set of finite G-sets that is defined as follows: a family of G-maps $\{\alpha_i : U_i \to X\}_{i \in I}$ is a covering if, for every $x \in X$, there exist an $i \in I$ and $u \in \alpha_i^{-1}(x)$ with the index $|\mathrm{Stab}_G(x) : \mathrm{Stab}_G(u)|$ being prime to p. ("Sipp" is an acronym for either "stabilisers of index prime to p" or "stabilisateurs d'indice premier à p".) We are interested in the cover $\mathcal{U} = \{G/S \to G/G\}$ and the constant sipp-sheaf $\mathbb{G}_m$ associated with the abelian group $k^\times$. More explicitly, $\mathbb{G}_m : G\text{-Set}^{\mathrm{op}} \to \mathrm{Ab}$ is the finite-coproduct-preserving functor that takes values

$$\mathbb{G}_m(G/H) \cong \begin{cases} k^\times & \text{if } p \text{ divides } |H| \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

all of whose restrictions are componentwise either the identity or zero. Balmer's obstruction lies in $\check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)$, and we recall its definition in Remark A.3.

A.1. **Lemma.** *Let $F : \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathrm{Ab}$ be a functor and let $\tilde{F} : G\text{-Set}^{\mathrm{op}} \to \mathrm{Ab}$ denote the right Kan extension of F along the opposite of the inclusion $i : \mathcal{O}_{\mathcal{S}}(G) \hookrightarrow G\text{-Set}$. There is a natural isomorphism*

$$\mathrm{H}^*(\mathcal{O}_{\mathcal{S}}(G);\, F) \cong \check{\mathrm{H}}^*(\mathcal{U};\, \tilde{F}).$$

A.2. *Remark.* If $F : \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathrm{Ab}$ is the constant functor with value $k^\times$, then $\tilde{F}$ is isomorphic to $\mathbb{G}_m$ and the lemma implies that $\mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G);\, k^\times) \cong \check{\mathrm{H}}^2(\mathcal{U};\, \mathbb{G}_m)$. To see this, note that for any G/H we have

$$\mathrm{Ran}_i(F)(G/H) \simeq \lim\big((i\,/\,(G/H))^{\mathrm{op}} \to \mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}} \to \mathrm{Ab}\big)$$

and that $i/(G/H) \simeq \mathcal{O}_{\mathcal{S}}(H)$. This means that if p does not divide $|H|$, then $\tilde{F}(G/H) = 0$, while if p divides $|H|$, then $\tilde{F}(G/H) \cong k^\times$ because $\mathcal{O}_{\mathcal{S}}(H)$ is a connected category. As $\tilde{F}$ also preserves finite coproducts, it is isomorphic to $\mathbb{G}_m$.

Proof of Lemma A.1. We can compute a Kan extension along i in two steps: we first extend to $G\text{-Set}_{\mathcal{S}}$, the full subcategory of G-Set on objects whose isotropy subgroups are all non-trivial p-groups, and from there to G-Set itself. The category $G\text{-Set}_{\mathcal{S}}$ is the free completion of $\mathcal{O}_{\mathcal{S}}(G)$ under finite coproducts, so by [JM92, Corollary 4.4] we have

$$\mathrm{H}^*(\mathcal{O}_{\mathcal{S}}(G);\, F) \cong \mathrm{H}^*(G\text{-Set}_{\mathcal{S}};\, \tilde{F}).$$

Observe that for every $n \geq 1$ the product $(G/S)^n$ exists in $G\text{-Set}_{\mathcal{S}}$, given by discarding all free orbits from the "ordinary" product $(G/S)^n$ taken in G-Set. This follows from the fact that there are no morphisms $G/P \to G/\{1\}$ for $P \in \mathcal{S}_p(G)$. We get a simplicial object

$$\cdots \;\substack{\to\\\to\\\to\\\to}\; \frac{G}{S}\times\frac{G}{S}\times\frac{G}{S} \;\substack{\to\\\to\\\to}\; \frac{G}{S}\times\frac{G}{S} \;\substack{\to\\\to}\; \frac{G}{S}$$

in $G\text{-Set}_{\mathcal{S}}$; write $X^\bullet : \Delta^{\mathrm{op}} \to G\text{-Set}_{\mathcal{S}}$ for the functor that classifies it.

Observe that every object Y in $G\text{-Set}_{\mathcal{S}}$ admits a map $Y \to G/S$, so pulling back along the functor $X^\bullet$ preserves homotopy colimits [MNN17, Proposition 6.28]. This implies that we have

$$\mathrm{H}^*(G\text{-Set}_{\mathcal{S}};\, \tilde{F}) \cong \mathrm{H}^*(\Delta;\, \tilde{F}\circ(X^\bullet)^{\mathrm{op}}).$$

It remains to identify $\mathrm{H}^*(\Delta;\, \tilde{F}\circ(X^\bullet)^{\mathrm{op}})$ with $\check{\mathrm{H}}^*(\mathcal{U};\, \tilde{F})$. To this end, let $A^\bullet : \Delta \to \mathrm{Ab}$ denote the cosimplicial abelian group $\tilde{F}\circ(X^\bullet)^{\mathrm{op}}$ and write $\mathbb{Z}$ for the constant cosimplicial abelian group with value $\mathbb{Z}$. The functor category $[\Delta, \mathrm{Ab}]$ is an abelian category [Mac98, p. 199] and we have

$$\mathrm{H}^*(\Delta;\, A^\bullet) \cong \mathrm{Ext}^*_{[\Delta,\mathrm{Ab}]}(\mathbb{Z}, A^\bullet).$$

Consider $\mathbb{Z}\Delta(-,-) : \Delta^{\mathrm{op}}\times\Delta \to \mathrm{Ab}$, the $\mathbb{Z}$-linearisation of the morphism sets in the category Δ. We claim that

$$\ldots \to \mathbb{Z}\Delta([2],-) \to \mathbb{Z}\Delta([1],-) \to \mathbb{Z}\Delta([0],-) \to \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ in $[\Delta, \mathrm{Ab}]$. Each of the functors $\mathbb{Z}\Delta([n],-)$ is projective, since Yoneda's lemma implies that $\mathrm{Hom}_{[\Delta,\mathrm{Ab}]}(\mathbb{Z}\Delta([n],-),-)$ preserves epimorphisms. We can check exactness pointwise: the chain complex

$$\ldots \to \mathbb{Z}\Delta([2],[m]) \to \mathbb{Z}\Delta([1],[m]) \to \mathbb{Z}\Delta([0],[m]) \to 0$$

identifies with the simplicial chain complex $\mathrm{C}_\bullet(\Delta^m)$, whose homology is concentrated in degree zero since Δ^m is contractible. Therefore we get a projective resolution as claimed.

Finally, we consider the cochain complex that computes Ext: by another application of Yoneda's lemma, the chain complex

$$0 \to \mathrm{Hom}_{[\Delta,\mathrm{Ab}]}(\mathbb{Z}\Delta([0],-), A^\bullet) \to \mathrm{Hom}_{[\Delta,\mathrm{Ab}]}(\mathbb{Z}\Delta([1],-), A^\bullet) \to \ldots$$

identifies with

$$0 \to A^0 \to A^1 \to A^2 \to \dots$$

and we observe that this complex is precisely the Čech complex $\check{C}^\bullet(\mathcal{U}; \tilde{F})$. □

A.3. *Remark.* We briefly recall the definition of the obstruction defined in [Bal15, Theorem 10.7]. We consider the cosimplicial diagram of ∞-categories

$$\mathrm{StMod}(G/S) \rightrightarrows \mathrm{StMod}((G/S)^2) \substack{\to\\\to\\\to} \dots$$

appearing in the proof of [Mat16, Proposition 9.13] (*c.f.* the definition of $\mathrm{StMod}(X)$ for a G-set X in Section 3). This is the counterpart on the categorical level of the cosimplicial abelian group $A^\bullet$; one can recover $A^\bullet$ from this diagram by composing with the functor that takes a symmetric monoidal ∞-category to the homotopy classes of automorphisms of its unit.

Let $M \in \mathrm{StMod}(G/S)$ be such that $d^0M \simeq d^1M$ in $\mathrm{StMod}((G/S)^2)$; this condition is precisely saying that the equivalence class of M is an element of $\check{\mathrm{H}}^0(\mathcal{U}; \mathrm{Pic}^{\mathrm{st}})$, in Balmer's notation. Choose an equivalence $\xi : d^0M \xrightarrow{\sim} d^1M$ and define $\zeta := (d^1\xi)^{-1} \circ d^2\xi \circ d^0\xi$. This is an automorphism of d^0d^0M, which we can canonically identify with an element of $\mathbb{G}_m((G/S)^3)$. The morphism ζ gives a 2-cocycle in $\check{\mathrm{C}}^2(\mathcal{U}; \mathbb{G}_m)$ whose equivalence class in $\check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)$ depends only on the equivalence class of M, not on the choice of ξ. This is Balmer's obstruction class.

A.4. **Lemma.** *Under the isomorphism given by Lemma A.1, the obstruction* $[\alpha] \in \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$ *defined in Theorem 4.5 identifies with the obstruction in* $[\zeta] \in \check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)$ *recalled in Remark A.3.*

Proof. We will trace $[\alpha]$ through the isomorphisms in Lemma A.1 and show that it agrees with $[\zeta]$. The first isomorphism in Lemma A.1 extended $[\alpha]$ to $[\alpha'] \in \mathrm{H}^2(G\text{-Set}_{\mathcal{S}}; \mathbb{G}_m)$. A G-stable endotrivial module M_S gives us an element of $\lim_{G\text{-Set}_{\mathcal{S}}^{\mathrm{op}}} \pi_0 \,\mathrm{Pic}\,\mathrm{StMod}(-)$, *i.e.* an element $M_X \in \mathrm{StMod}(X)$ for every finite G-set X, such that $f^*(M_Y) \simeq M_X$ for every morphism $f : X \to Y$ in $G\text{-Set}_{\mathcal{S}}$. (Note that although we have made a choice of representative M_X here, the equivalence class of M_X is canonical.) For every such f, we choose a specific equivalence $\lambda_f : f^*(M_Y) \xrightarrow{\sim} M_X$. The 2-cocycle α' is then given by sending

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

to the element of $\mathbb{G}_m(X)$ determined by the automorphism

$$f^*g^*M_Z \xrightarrow{\lambda_{gf}} M_X \xleftarrow{\lambda_f} f^*M_Y \xleftarrow{f^*\lambda_g} f^*g^*M_Z$$

since this restricts to the correct class $[\alpha] \in \mathrm{H}^2(\mathcal{O}_{\mathcal{S}}(G); k^\times)$. The class $[\alpha'] \in \mathrm{H}^2(G\text{-Set}; \mathbb{G}_m)$ depends only on the equivalence class of M_S in StMod_{kS}, not on M_S itself, nor on the choices of M_X and λ_f.

The next isomorphism in Lemma A.1 was induced by restriction along the simplicial object $X^\bullet : \Delta^{\mathrm{op}} \to G\text{-Set}_{\mathcal{S}}$ given by

$$\cdots \substack{\to\\\to\\\to\\\to} \frac{G}{S} \times \frac{G}{S} \times \frac{G}{S} \substack{\to\\\to\\\to} \frac{G}{S} \times \frac{G}{S} \rightrightarrows \frac{G}{S}.$$

Let α'' denote the restriction of α' along $X^\bullet$. We can describe $\alpha'' \in \mathrm{C}^2(\Delta; A)$ by choosing representatives $M_{[n]} \in \mathrm{StMod}((G/S)^{n+1})$ and equivalences $\lambda_\theta : \theta_*(M_{[n]}) \xrightarrow{\sim} M_{[m]}$ for every $\theta : [n] \to [m]$; the resulting cocycle α'' is given by sending

$$[n] \xrightarrow{\theta} [m] \xrightarrow{\psi} [l]$$

to the element of $\mathbb{G}_m((G/S)^{l+1})$ determined by the automorphism

$$\psi_*\theta_*M_{[n]} \xrightarrow{\psi_*\lambda_\theta} \psi_*M_{[m]} \xrightarrow{\lambda_\psi} M_{[l]} \xleftarrow{\lambda_{\psi\theta}} \psi_*\theta_*M_{[n]} \tag{A.5}$$

of $\psi_*\theta_*M_{[n]}$. As before, the cohomology class $[\alpha'']$ is independent of all the choices we made.

The third and final isomorphism of Lemma A.1 sent $\mathrm{H}^*(\Delta;\, A)$ to $\mathrm{H}^*(A)$, which amounts to removing a subdivision. The cochain complexes $A^\bullet$ and $\mathrm{C}^\bullet(\Delta;\, A)$ arise respectively from the projective resolutions of $\mathbb{Z} \in [\Delta, \mathrm{Ab}]$ given by $\mathbb{Z}\Delta([\bullet], -)$ and $\mathrm{C}_\bullet(\Delta/-)$. We can define a comparison map between these resolutions by sending the identity in $\mathbb{Z}\Delta([n],[n])$ to the alternating sum

$$\sum(-1)^{i_1+\ldots+i_n}[0] \xrightarrow{d^{i_1}} [1] \xrightarrow{d^{i_2}} \ldots \xrightarrow{d^{i_n}} [n] \xrightarrow{\mathrm{id}} [n] \tag{A.6}$$

in $\mathrm{C}_n(\Delta/[n])$. The sum is indexed over all possible choices of $(i_1, i_2, \ldots, i_n)$. The map in (A.6) determines the natural transformation in every degree by Yoneda's lemma, and the fact that it is a chain map follows from the cosimplicial identity $d^j \circ d^i = d^i \circ d^{j-1}$ for $i < j$. Since this natural transformation lives over the identity $\mathbb{Z} \to \mathbb{Z}$, it is the unique-up-to-homotopy quasi-isomorphism between the resolutions.

We deduce that the image in A^2 of $\alpha'' \in \mathrm{C}^2(\Delta;\, A)$ under the above quasi-isomorphism is

$$\sum(-1)^{i_1+i_2}\alpha''([0] \xrightarrow{d^{i_1}} [1] \xrightarrow{d^{i_2}} [2]).$$

We can draw this alternating sum in a diagram:

$$\begin{array}{ccccccccc}
d^0d^0M_{[0]} & \dashrightarrow & d^0d^1M_{[0]} = d^2d^0M_{[0]} & \dashrightarrow & d^2d^1M_{[0]} = d^1d^1M_{[0]} & \dashleftarrow & d^1d^0M_{[0]} \\
{\scriptstyle d^0\lambda_{d^0}}\searrow & & \swarrow{\scriptstyle d^0\lambda_{d^1}} \quad {\scriptstyle d^2\lambda_{d^0}}\searrow & & \swarrow{\scriptstyle d^2\lambda_{d^1}} \quad {\scriptstyle d^1\lambda_{d^1}}\searrow & & \swarrow{\scriptstyle d^1\lambda_{d^0}} \\
 & d^0M_{[1]} & & d^2M_{[1]} & & d^1M_{[1]} & \\
 & \downarrow{\scriptstyle \lambda_{d^0}} & & \downarrow{\scriptstyle \lambda_{d^2}} & & \downarrow{\scriptstyle \lambda_{d^1}} & \\
 & M_{[2]} & & M_{[2]} & & M_{[2]} &
\end{array}$$

Here the solid arrows come from the terms in the alternating sum, expanded out using the description of α'' in (A.5); there are six maps omitted, namely the backwards-facing arrows in (A.5), which cancel out in pairs. The alternating sum itself is equal to the automorphism of $d^0d^0M_{[0]} = d^1d^0M_{[0]}$ given by the dashed maps. If we choose ξ to be given by

$$d^0M_{[0]} \xrightarrow{\lambda_{d^0}} M_{[1]} \xleftarrow{\lambda_{d^1}} d^1M_{[0]}$$

then these dashed maps are exactly the maps $d^0(\xi)$, $d^2(\xi)$, and $d^1(\xi)$. Comparing with Remark A.3, we see that the image of the class $[\alpha'']$ is equal to $[\zeta]$. □

A.7. *Remark.* Under the identification of Lemma A.4, Balmer's [Bal15, Theorem 10.7(e)] is equivalent to the exactness at $\lim_{\mathcal{O}_S(G)^{\mathrm{op}}} T(-)$ of the sequence appearing in Theorem 4.5.

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Part III

Decompositions of the stable module ∞-category

DECOMPOSITIONS OF THE STABLE MODULE ∞-CATEGORY

JOSHUA HUNT

Abstract. We show that the stable module ∞-category of a finite group G decomposes in three different ways as a limit of the stable module ∞-categories of certain subgroups of G. Analogously to Dwyer's terminology for homology decompositions, we call these the centraliser, normaliser, and subgroup decompositions. We construct centraliser and normaliser decompositions and extend the subgroup decomposition (constructed by Mathew) to more collections of subgroups. The key step in the proof is extending the stable module ∞-category to be defined for any G-space, then showing that this extension only depends on the S-equivariant homotopy type of a G-space. The methods used are not specific to the stable module ∞-category, so may also be applicable in other settings where an ∞-category depends functorially on G.

1. Introduction

Let G be a finite group and p be a prime dividing the order of G. A homology decomposition of the classifying space $\mathrm{B}G$ is a diagram of spaces $F : D \to \mathcal{S}$ such that, for every $d \in D$, the space $F(d)$ has the homotopy type of $\mathrm{B}H$ for some $H \leq G$, together with a map

$$\operatorname{hocolim} F \to \mathrm{B}G$$

that induces an isomorphism on mod p homology. Homology decompositions have a long history in algebraic topology, with an early success being their use in [JMO92] to classify self-maps of classifying spaces of compact, connected, simple Lie groups. They also played an important role in the classification of p-compact groups; see [Gro10] for a survey. More recently, similar decomposition techniques have found applications in modular representation theory, for example in [Mat16], [Gro18], and [BGH].

In [Dwy97], Dwyer was able to give a unified treatment of three different types of homology decompositions for a fixed collection $\mathscr{C}$ of subgroups of G (where by the term *collection* we always mean a set of subgroups that is closed under conjugation by elements of G):

Subgroup: let $D = \mathcal{O}_{\mathscr{C}}(G)$, the orbit category, and let $F : \mathcal{O}_{\mathscr{C}}(G) \to \mathcal{S}$ take G/H to $\mathrm{B}H$.
Centraliser: let $D = \mathcal{F}_{\mathscr{C}}(G)$, the fusion category, and let $F : \mathcal{F}_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}$ take H to $\mathrm{B}C_G(H)$.
Normaliser: let $D = \bar{s}S_{\mathscr{C}}(G)$, the orbit simplex category, and let $F : \bar{s}S_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}$ take a simplex $\sigma = (H_0 < \ldots < H_n)$ to $\mathrm{B}N_G(\sigma)$, where $N_G(\sigma)$ denotes $\bigcap_{0 \leq i \leq n} N_G(H_i)$.

We recall the definition of the indexing categories below, in Section 2. Dwyer showed that in all of these cases, F provides a mod p homology decomposition of G if and only if the natural map $\mathscr{C}_{\mathrm{hG}} \to (*)_{\mathrm{hG}} \simeq \mathrm{B}G$ induces an isomorphism on mod p homology. Dwyer calls a collection $\mathscr{C}$ satisfying this condition *ample*.

The stable module category StMod_{kG} of G over a field k of characteristic p is obtained by "quotienting" the module category Mod_{kG} by the projective modules. In this paper, we show that analogues of the subgroup, centraliser and normaliser decompositions exist for StMod_{kG},

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viewed as an ∞-category, describing it in three different ways as a limit of ∞-categories StMod_{kH} for subgroups $H \leq G$. In this setting, Mathew has already shown the existence of the subgroup decomposition for certain collections:

Theorem ([Mat16, Corollary 9.16]). *Let $\mathscr{C}$ be a collection of subgroups of G that is closed under intersection and such that every elementary abelian p-subgroup of G is contained in a subgroup in $\mathscr{C}$. There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{G/H \in \mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kH}.$$

Note that the change from a homotopy colimit (in Dwyer's result) to a homotopy limit (in Mathew's result) is due to the differing variances of the homotopy orbits functor $(-)_{\mathrm{h}G}$ and the stable module ∞-category functor $\mathrm{StMod}(-)$. Following ideas of Dwyer and others, we use G-spaces to encode decompositions of $\mathrm{StMod}(-)$: we formally Kan extend the functor

$$\mathrm{StMod}(-) : \mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$$

to a functor defined on any G-space

$$\mathrm{StMod}(-) : \mathcal{S}_G^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}. \tag{1.1}$$

This extended functor takes small homotopy colimits of G-spaces to homotopy limits of ∞-categories, so for any diagram of G-spaces $F : D \to \mathcal{S}_G$, the canonical map $\mathrm{hocolim}(F) \to *$ induces a comparison map

$$\mathrm{StMod}_{kG} \simeq \mathrm{StMod}(*) \to \lim_{D^{\mathrm{op}}} \mathrm{StMod}(F(d)). \tag{1.2}$$

Dwyer constructed G-spaces (depending on the collection $\mathscr{C}$) that encode the three homology decompositions listed above. For convenience, we will temporarily refer to these G-spaces as *encoding G-spaces*. Applying $\mathrm{StMod}(-)$ to the encoding G-spaces for $\mathscr{C}$ gives functors as in (1.2) that are candidates for the three decompositions of the stable module ∞-category.

To show that we do get a decomposition of the stable module ∞-category, we need to prove that the comparison functor (1.2) is an equivalence. For this we use work of Grodal–Smith [GS06], which lists cases when certain canonical maps between the encoding G-spaces are S-equivalences, where S is a Sylow p-subgroup. (Recall that that a map $f : X \to Y$ of G-spaces is an S-equivalence if it induces a homotopy equivalence $X^P \xrightarrow{\sim} Y^P$ on P-fixed points for every p-subgroup $P \leq G$.) In Section 5, we prove that the extended functor (1.1) inverts S-equivalences. In fact, we prove a more general result (Theorem 4.8):

Theorem A. *Let $\mathcal{A}$ and $\mathcal{B}$ be small ∞-categories, $\mathcal{C}$ be an ∞-category with all small colimits, and*

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

be functors with i fully faithful. Let $\tilde{F} : \mathcal{P}(\mathcal{B}) \to \mathcal{C}$ denote the left Kan extension of F along the Yoneda embedding $y_{\mathcal{B}}$. There is a factorisation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{B}) & \xrightarrow{\tilde{F}} & \mathcal{C} \\ {\scriptstyle i^*}\downarrow & \nearrow_{\tilde{F}'} & \\ \mathcal{P}(\mathcal{A}) & & \end{array}$$

if and only if F is the left Kan extension of its restriction to $\mathcal{A}$, i.e. *the natural map*

$$\mathrm{colim}(i/b \to \mathcal{A} \xrightarrow{Fi} \mathcal{C}) \to F(b)$$

is an equivalence for every $b \in \mathcal{B}$.

From this criterion, we deduce Theorem 5.3:

Theorem B. *The right Kan extension of* StMod(−) : $\mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ *along the opposite of the Yoneda embedding* $\mathcal{O}(G) \to \mathcal{S}_G$ *factors through the restriction map*

$$\mathcal{S}_G^{\mathrm{op}} \simeq \mathcal{P}(\mathcal{O}(G))^{\mathrm{op}} \to \mathcal{P}(\mathcal{O}_p(G))^{\mathrm{op}}.$$

In particular, StMod(−) *sends* S*-equivalences in* $\mathcal{S}_G$ *to equivalences of* ∞*-categories.*

We note that the only property of StMod(−) used to deduce Theorem B from Theorem A is the existence of a subgroup decomposition, as in Mathew's result above. The approach therefore applies whenever an ∞-category depends functorially on G and satisfies an analogous descent condition.

Given Theorem B, it is enough to find a zig-zag of S-equivalences from the encoding G-space for Mathew's subgroup decomposition to the encoding G-space for the decomposition that we are interested in. This problem was studied by Grodal–Smith in [GS06] and we use their results in Section 6 to obtain our main theorem (Theorem 6.4):

Theorem C. *Let* $\mathscr{C}$ *be one of the collections* $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{I}_p(G)$, *or* $\mathcal{Z}_p(G)$. *There is a subgroup decomposition*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{G/P \in \mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP}$$

and a normaliser decomposition

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{[\sigma] \in \bar{s}S_{\mathscr{C}}(G)} \mathrm{StMod}_{kN_G(\sigma)}.$$

If $\mathscr{C}$ *is* $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, *or* $\mathcal{Z}_p(G)$, *then there is additionally a centraliser decomposition*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{P \in \mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}_{kC_G(P)}.$$

The collections mentioned in the theorem are defined as follows:

(i) $\mathcal{S}_p(G)$ is the collection of non-trivial p-subgroups of G,
(ii) $\mathcal{A}_p(G)$ is the collection of non-trivial elementary abelian p-subgroups,
(iii) $\mathcal{B}_p(G)$ is the collection of non-trivial p-radical subgroups, *i.e.* non-trivial p-subgroups $P \leq G$ such that P is the maximal normal p-subgroup in $N_G(P)$,
(iv) $\mathcal{I}_p(G)$ is the collection of all non-trivial p-subgroups that are the intersection of a set of Sylow p-subgroups, and
(v) $\mathcal{Z}_p(G)$ is the subcollection of $\mathcal{A}_p(G)$ consisting of those V such that V is the set of elements in the centre of $C_G(V)$ whose order divides p, *i.e.* such that $V = \Omega_1 O_p Z(C_G(V))$, using standard group-theoretic notation such as found in [Asc00].

1.3. *Remark.* Neither $\mathcal{B}_p(G)$ nor $\mathcal{Z}_p(G)$ are closed under intersections, so the subgroup decompositions for $\mathcal{B}_p(G)$ and $\mathcal{Z}_p(G)$ in Theorem 6.4 are new and do not follow immediately from Mathew's subgroup decomposition. For example, if $G = \mathrm{PSL}_3(7)$, then there are rank two elementary abelian subgroups

$$\begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

that are contained in both $\mathcal{B}_p(G)$ and $\mathcal{Z}_p(G)$, but whose intersection

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

is contained in neither collection.

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2. Notation, conventions, and background

Throughout the paper, G will refer to a finite group and p will be a fixed prime dividing the order of G. We let k be a field of characteristic p. A *collection* of subgroups of G is a set of subgroups that is closed under conjugation by elements of G.

We let $\mathcal{O}(G)$ denote the *orbit category* of G, whose objects are transitive left G-sets and whose morphisms are G-equivariant maps between them. For any collection $\mathscr{C}$ of subgroups of G, we let $\mathcal{O}_{\mathscr{C}}(G)$ denote the full subcategory of $\mathcal{O}(G)$ on those G-sets whose isotropy subgroups are contained in $\mathscr{C}$. Note that every object of $\mathcal{O}(G)$ is isomorphic to a G-set of the form G/H, with H a subgroup of G, and that G/H lies in $\mathcal{O}_{\mathscr{C}}(G)$ if and only if $H \in \mathscr{C}$. With this identification, the morphisms in the orbit category are related to subconjugation relations between subgroups:

$$\mathrm{Hom}_{\mathcal{O}(G)}(G/H, G/K) \;\cong\; \{\, g \in G \,:\, H^g \leq K \,\}/K.$$

When $\mathscr{C}$ is the collection of all p-subgroups of G, including the trivial subgroup, we will use the notation $\mathcal{O}_p(G)$ instead of $\mathcal{O}_{\mathscr{C}}(G)$.

We let $\mathcal{F}_{\mathscr{C}}(G)$ denote the *fusion category* of G, whose objects are the subgroups in $\mathscr{C}$ and whose morphisms are homomorphisms that are induced by conjugation by an element of G.

We let $\bar{s}S_{\mathscr{C}}(G)$ denote the *orbit simplex category* of G, which is the poset of G-conjugacy classes of non-empty chains $\sigma = (H_0 < \ldots < H_n)$ of subgroups in $\mathscr{C}$, ordered by refinement: that is, we have $[\sigma] \leq [\tau]$ if we can find representatives σ and τ for the conjugacy classes such that $\sigma \subseteq \tau$. The objects of $\bar{s}S_{\mathscr{C}}(G)$ identify with the G-conjugacy classes of non-degenerate simplices of the nerve of $\mathscr{C}$.

Since we deal exclusively with homotopy (co)limits, we will drop the adjective "homotopy" here: when we refer to a "colimit of G-spaces" we will implicitly mean a homotopy colimit. We follow Lurie's convention of using the prefix "∞-" instead of "$(\infty, 1)$-". We use $\widehat{\mathrm{Cat}}_\infty$ to denote the ∞-category of large ∞-categories, which has all large limits (though we will only need the existence of small limits). Let $\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors that preserve small colimits.

The *stable module ∞-category* StMod_{kG} is defined as the localisation of the module category Mod_{kG} at the *stable equivalences*, *i.e.* at those maps $f : M \to N$ and $g : N \to M$ such that $fg - \mathrm{id}_N$ and $gf - \mathrm{id}_M$ both factor through a projective module. The homotopy category of the stable module ∞-category has been studied by representation theorists: for example, Benson–Iyengar–Krause [BIK11] use it as a way of classifying kG-modules when Mod_{kG} has wild representation type. The objects of the homotopy category are kG-modules and the hom sets are given by

$$\pi_0 \,\mathrm{Map}_G(M, N) \cong \mathrm{Hom}_G(M, N)/(f \sim 0 \text{ if } f \text{ factors through a projective}).$$

The stable module ∞-category is a presentable, stable, symmetric monoidal ∞-category. See [Car96, Section 5] for a discussion of the homotopy category and [Mat15, Definition 2.2] for a construction of StMod_{kG} as an ∞-category.

In [Mat16, Section 9.5], Mathew constructs a functor

$$\mathrm{StMod}(-) : \mathcal{O}(G)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$$

whose value on G/H is equivalent to StMod_{kH}. Here $\mathrm{Pr}^{\mathrm{L,st}}$ denotes the ∞-category of presentable, stable ∞-categories and left adjoint functors between them. This functor will play a crucial role for us, so we spend the rest of this section describing its construction. For any $H \leq G$, we have a symmetric monoidal restriction functor $\mathrm{res}^G_H : \mathrm{StMod}_{kG} \to \mathrm{StMod}_{kH}$, whose right adjoint $\mathrm{coind}^G_H : \mathrm{StMod}_{kH} \to \mathrm{StMod}_{kG}$ is consequently lax symmetric monoidal. This implies that $\mathrm{coind}^G_H(k)$ is a commutative algebra object of StMod_{kG}, which we will denote A^G_H. The underlying module of A^G_H is $\prod_{G/H} k$ with its permutation action.

In [Bal15, Theorem 1.2], Balmer proves that the homotopy category of StMod_{kH} is equivalent to the category of modules over A^G_H internal to the homotopy category of StMod_{kG}. Proposition 9.12 of [Mat16] generalises this result to the ∞-category StMod_{kH}:

2.1. **Theorem** (Balmer, Mathew)**.** *There is a natural symmetric monoidal equivalence*

$$\mathrm{StMod}_{kH} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A^G_H)$$

induced by coinduction, under which the free/forget adjunction $\mathrm{StMod}_{kG} \leftrightarrows \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A^G_H)$ *corresponds to the restriction/coinduction adjunction* $\mathrm{StMod}_{kG} \leftrightarrows \mathrm{StMod}_{kH}$*.*

We have a functor

$$\begin{array}{rcc} \mathcal{O}(G)^{\mathrm{op}} & \to & \mathrm{CAlg}(\mathrm{StMod}_{kG}) \\ G/H & \mapsto & A^G_H \end{array}$$

that on underlying modules sends a morphism $G/H \to G/K$ to the pullback map $\prod_{G/K} k \to \prod_{G/H} k$. This functor can be constructed by composing the analogous functor $\mathcal{O}(G)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Mod}_{kG})$ with the localisation functor $\mathrm{Mod}_{kG} \to \mathrm{StMod}_{kG}$. Theorem 2.1 implies that we obtain a functor

$$\mathrm{StMod}(-) : \mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

that takes G/H to an ∞-category equivalent to StMod_{kH}. By using the description given in [MNN17, Construction 5.23] of the inverse to the equivalence in Theorem 2.1, one can check that a morphism $G/H \to G/K$ in $\mathcal{O}(G)$ is sent to the restriction functor $\mathrm{StMod}_{kK} \to \mathrm{StMod}_{kH}$. Since both StMod_{kH} and $\mathrm{res}^K_H : \mathrm{StMod}_{kK} \to \mathrm{StMod}_{kH}$ lie in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$, we have constructed a functor

$$\mathcal{O}(G)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$$

as desired.

3. THE STABLE MODULE ∞-CATEGORY OF A G-SPACE

Let $\mathcal{S}$ denote the ∞-category of small spaces. Recall that the ∞-category $\mathcal{S}_G$ of small G-spaces is equivalent to $\mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathcal{S})$ and the Yoneda embedding $y : \mathcal{O}(G) \to \mathcal{S}_G$ identifies with the inclusion of $\mathcal{O}(G)$ as the transitive, discrete G-spaces.

The stable module ∞-category determines a functor $\mathrm{StMod}(-) : \mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ that takes G/H to StMod_{kH}. We can formally extend this functor to G-spaces by right Kan extension along the opposite of the Yoneda embedding:

$$\begin{array}{ccc} \mathcal{O}(G)^{\mathrm{op}} & \longrightarrow & \widehat{\mathrm{Cat}}_\infty \\ {\scriptstyle y}\downarrow & \nearrow_{\mathrm{StMod}(-)} & \\ \mathcal{S}_G^{\mathrm{op}} & & \end{array}$$

Since $\mathcal{O}(G)$ is small, the existence of small limits in $\widehat{\mathrm{Cat}}_\infty$ guarantees the existence of the Kan extension, which we still denote by StMod(−). It sends small (homotopy) colimits of G-spaces to limits in $\widehat{\mathrm{Cat}}_\infty$.

3.1. *Remark.* We could equally well have taken StMod(−) to be a functor to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$ while carrying out the above construction, since $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$ also has small limits and the composition $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}}) \to \mathrm{Pr}^{\mathrm{L,st}} \to \widehat{\mathrm{Cat}}_\infty$ preserves limits [Lur17, Corollary 3.2.2.4]. In other words, the "stable module ∞-category of a G-space" is presentable, stable, and has a symmetric monoidal structure that preserves finite colimits in each variable. We will not need these facts, but mention them to point out that the extended definition of the stable module ∞-category shares many features with the standard definition.

4. FACTORISING KAN EXTENSIONS

Our first goal is to show that StMod(−) only sees the S-equivariant homotopy type of a G-space: that is, we have a factorisation

$$\begin{array}{ccc} \mathcal{S}_G^{\mathrm{op}} \simeq \mathcal{P}(\mathcal{O}(G))^{\mathrm{op}} & \longrightarrow & \widehat{\mathrm{Cat}}_\infty \\ \downarrow{\scriptstyle i^*} & \nearrow & \\ \mathcal{P}(\mathcal{O}_p(G))^{\mathrm{op}} & & \end{array}$$

where $i\colon \mathcal{O}_p(G) \hookrightarrow \mathcal{O}(G)$ is the inclusion of the full subcategory of transitive G-sets with p-group isotropy and $\mathcal{P}(\mathcal{A})$ is the ∞-category $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ of presheaves on $\mathcal{A}$.

For simplicity of notation, we dualise and consider the following question:

4.1. *Question.* Let $\mathcal{A}$ and $\mathcal{B}$ be small ∞-categories, $\mathcal{C}$ be an ∞-category with all small colimits, and

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

be functors with i fully faithful. When does the left Kan extension $\tilde{F}$ of F along the Yoneda embedding $y_{\mathcal{B}}$ admit a factorisation through $\mathcal{P}(\mathcal{A})$ as indicated in the diagram below?

$$\begin{array}{ccc} \mathcal{B} & & \\ {\scriptstyle y_{\mathcal{B}}}\downarrow & \searrow^{F} & \\ \mathcal{P}(\mathcal{B}) & \xrightarrow{\tilde{F}} & \mathcal{C} \\ {\scriptstyle i^*}\downarrow & \nearrow_{\tilde{F}'} & \\ \mathcal{P}(\mathcal{A}) & & \end{array}$$

We answer this question in Theorem 4.8, providing a necessary and sufficient condition on F for such a factorisation $\tilde{F}'$ to exist: namely, F must be equivalent to the left Kan extension of Fi along i.

4.2. **Notation.** In this situation, there are two functors that are induced by restriction along i (or its opposite) and hence could reasonably be denoted by i^*, namely

$$\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{A}) \quad \text{and} \quad \mathrm{Fun}(\mathcal{B},\mathcal{C}) \to \mathrm{Fun}(\mathcal{A},\mathcal{C}).$$

We will denote the first functor by i^* and the second functor instead by j^*. We hope that this prevents more confusion than it causes. Both of these functors have left adjoints given by left Kan extension, which we will write as

$$i_! : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B}) \quad \text{and} \quad j_! : \mathrm{Fun}(\mathcal{A},\mathcal{C}) \to \mathrm{Fun}(\mathcal{B},\mathcal{C}).$$

We will similarly use $(y_{\mathcal{B}})_!$ to denote left Kan extension along the Yoneda embedding. The adjunction $i_! \dashv i^*$ induces an adjunction

$$\begin{array}{c} \mathrm{Fun}(\mathcal{P}(\mathcal{A}),\mathcal{C}) \\ (i^*)^* \big\downarrow \dashv \big\uparrow (i_!)^* \\ \mathrm{Fun}(\mathcal{P}(\mathcal{B}),\mathcal{C}). \end{array}$$

4.3. *Remark.* We briefly summarise the proof of Theorem 4.8, making forward reference to lemmas that we will prove later in the section. We will show that the following statements are all equivalent:

(i) $\tilde{F}$ factors through i^*.
(ii) The natural transformation $\tilde{F}i_!i^* \to \tilde{F}$ induced by the counit of the $i_! \dashv i^*$ adjunction is an equivalence.
(iii) $\tilde{F}$ is equivalent, via the counit of the $(i^*)^* \dashv (i_!)^*$ adjunction, to the composition $(i^*)^* \circ (i_!)^* \circ (y_{\mathcal{B}})_!$ applied to F.
(iv) $\tilde{F}$ is equivalent, via the counit of the $j_! \dashv j^*$ adjunction, to the composition $(y_{\mathcal{B}})_! \circ j_! \circ j^*$ applied to F.
(v) The natural transformation $j_!j^*F \to F$ induced by the counit of the $j_! \dashv j^*$ adjunction is an equivalence.

The equivalence of (i) and (ii) is Lemma 4.4. Since $\tilde{F}$ is the left Kan extension of F along $y_{\mathcal{B}}$, we see that (iii) is just a rewriting of (ii) with different notation. Lemmas 4.5 and 4.6 show that (iii) is equivalent to (iv). Checking that the natural transformation in (iv) is still induced by the counit is a straightforward but tiring diagram chase that we include in Appendix A. Finally, [Lur09, Theorem 5.1.5.6] shows that restriction along the Yoneda embedding induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{B}),\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{B},\mathcal{C}),$$

so (iv) is equivalent to (v).

The rest of the section fills in the details omitted in Remark 4.3. We begin by showing that if $\tilde{F}$ does factor through i^*, then such a factorisation is unique.

4.4. **Lemma.** *Let $H : \mathcal{P}(\mathcal{B}) \to \mathcal{C}$ be a functor. Any functor $\overline{H} : \mathcal{P}(\mathcal{A}) \to \mathcal{C}$ that satisfies $H \simeq \overline{H} \circ i^*$ must be given by $\overline{H} \simeq H \circ i_!$. Furthermore, H factors through i^* if and only if the natural transformation*

$$H\varepsilon : Hi_!i^* \to H$$

induced by the counit of the $i_! \dashv i^*$ adjunction is an equivalence in $\mathrm{Fun}(\mathcal{P}(\mathcal{B}),\mathcal{C})$.

Proof. Since i is fully faithful, the unit of the adjunction induces an equivalence $\mathrm{id} \xrightarrow{\sim} i^*i_!$. Therefore, $Hi_! \simeq \overline{H}i^*i_! \simeq \overline{H}$. The second claim is a straightforward check using naturality of the equivalence $H \simeq \overline{H} \circ i^*$ and the triangle identities for $i_! \dashv i^*$. □

The condition in Lemma 4.4 for a factorisation to exist is in terms of $\tilde{F}$, so our next goal is to rewrite this as a condition on F. We will need two lemmas regarding properties of Kan extensions.

4.5. **Lemma.** *There is a canonical equivalence*

$$Fi \xrightarrow{\sim} \tilde{F} i_! y_{\mathcal{A}},$$

which by the universal property of left Kan extension induces a natural transformation

$$\mathrm{Lan}_{y_{\mathcal{A}}}(Fi) \to \tilde{F} i_!$$

as functors $\mathcal{P}(\mathcal{A}) \to \mathcal{C}$. This natural transformation is an equivalence; that is,

$$(y_{\mathcal{A}})_! \circ j^* \xrightarrow{\sim} (i_!)^* \circ (y_{\mathcal{B}})_!$$

as functors $\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{A}), \mathcal{C})$.

Proof. Since $y_{\mathcal{B}}$ is fully faithful, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle y_{\mathcal{A}}}\downarrow & & {\scriptstyle y_{\mathcal{B}}}\downarrow & \nearrow{\scriptstyle \tilde{F} := \mathrm{Lan}_{y_{\mathcal{B}}}(F)} & \\ \mathcal{P}(\mathcal{A}) & \xrightarrow{i_!} & \mathcal{P}(\mathcal{B}) & & \end{array}$$

that gives rise to the first equivalence.

The three functors $\tilde{F}$, $i_!$, and $\mathrm{Lan}_{y_{\mathcal{A}}}(Fi)$ all preserve small colimits, so it is enough to check that they restrict along the Yoneda embedding $y_{\mathcal{A}}$ to equivalent functors in $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$, by [Lur09, Theorem 5.1.5.6]. We then observe that $\mathrm{Lan}_{y_{\mathcal{A}}}(Fi)$ restricts to Fi, because $y_{\mathcal{A}}$ is also fully faithful. □

4.6. **Lemma.** *Let $H : \mathcal{A} \to \mathcal{C}$ be a functor. There is a natural equivalence*

$$\mathrm{Lan}_{y_{\mathcal{A}}}(H) \circ i^* \simeq \mathrm{Lan}_{y_{\mathcal{B}} i}(H)$$

as functors $\mathcal{P}(\mathcal{B}) \to \mathcal{C}$. That is,

$$(i^*)^* \circ (y_{\mathcal{A}})_! \simeq (y_{\mathcal{B}})_! \circ j_!$$

as functors $\mathrm{Fun}(\mathcal{A}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{B}), \mathcal{C})$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ {\scriptstyle y_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle y_{\mathcal{B}}} \\ \mathcal{P}(\mathcal{A}) & \xrightarrow{i_!} & \mathcal{P}(\mathcal{B}) \end{array}$$

that induces a commutative diagram of pullback functors:

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{A}, \mathcal{C}) & \xleftarrow{j^*} & \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \\ \uparrow{\scriptstyle (y_{\mathcal{A}})^*} & & \uparrow{\scriptstyle (y_{\mathcal{B}})^*} \\ \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{A}), \mathcal{C}) & \xleftarrow{(i_!)^*} & \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{B}), \mathcal{C}) \end{array} \tag{4.7}$$

The adjunction $i_! \dashv i^*$ induces an adjunction $(i^*)^* \dashv (i_!)^*$, so all of the functors in the above diagram have left adjoints. Thus, we obtain another commutative diagram:

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{A},\mathcal{C}) & \xrightarrow{j_!} & \mathrm{Fun}(\mathcal{B},\mathcal{C}) \\ \downarrow{\scriptstyle (y_{\mathcal{A}})_!} & & \downarrow{\scriptstyle (y_{\mathcal{B}})_!} \\ \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{A}),\mathcal{C}) & \xrightarrow{(i^*)^*} & \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{B}),\mathcal{C}) \end{array}$$

This is what we aimed to prove. □

Combining the above lemmas, we deduce:

4.8. **Theorem.** *Let $\mathcal{A}$ and $\mathcal{B}$ be small ∞-categories, $\mathcal{C}$ be an ∞-category with all small colimits, and*

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

be functors with i fully faithful. Let $\tilde{F} : \mathcal{P}(\mathcal{B}) \to \mathcal{C}$ denote the left Kan extension of F along the Yoneda embedding $y_{\mathcal{B}}$. There is a factorisation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{B}) & \xrightarrow{\tilde{F}} & \mathcal{C} \\ \downarrow{\scriptstyle i^*} & \nearrow{\scriptstyle \tilde{F}'} & \\ \mathcal{P}(\mathcal{A}) & & \end{array}$$

if and only if F is the left Kan extension of its restriction to $\mathcal{A}$, i.e. *the natural map*

$$\mathrm{colim}(i/b \to \mathcal{A} \xrightarrow{Fi} \mathcal{C}) \;\to\; F(b)$$

is an equivalence for every $b \in \mathcal{B}$.

Proof. In Lemma 4.4 we showed that $\tilde{F}$ factors through i^* if and only if the natural transformation

$$\tilde{F} i_! i^* \to \tilde{F}$$

induced by the counit of the $i_! \dashv i^*$ adjunction is an equivalence. By Lemma 4.5, the left-hand side of this is equivalent to $\mathrm{Lan}_{y_{\mathcal{A}}}(Fi) \circ i^*$, while by Lemma 4.6, this in turn is equivalent to $\mathrm{Lan}_{y_{\mathcal{B}} i}(Fi)$. Since both sides of

$$\mathrm{Lan}_{y_{\mathcal{B}} i}(Fi) \to \tilde{F} \tag{4.9}$$

preserve colimits in $\mathcal{P}(\mathcal{B})$, we can check whether (4.9) is an equivalence after restricting along $y_{\mathcal{B}}$. Therefore, $\tilde{F}$ factors through i^* if and only if the natural transformation

$$\mathrm{Lan}_i(Fi) \to F$$

is an equivalence. □

5. THE STABLE MODULE ∞-CATEGORY IS S-HOMOTOPY INVARIANT

We can now return to the specific case that interests us, namely

$$\mathcal{O}_p(G)^{\mathrm{op}} \xrightarrow{i} \mathcal{O}(G)^{\mathrm{op}} \xrightarrow{\mathrm{StMod}(-)} \widehat{\mathrm{Cat}}_\infty.$$

We wish to show that we have an induced functor $\tilde{F}' : \mathcal{P}(\mathcal{O}_p(G))^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. By the dual of Theorem 4.8, this happens if and only if the natural map

$$\mathrm{StMod}(G/H) \ \to \ \lim\left((i/(G/H))^{\mathrm{op}} \to \mathcal{O}_p(G)^{\mathrm{op}} \xrightarrow{\mathrm{StMod}} \widehat{\mathrm{Cat}}_\infty\right)$$

is an equivalence for every $H \leq G$. The slice category $i/(G/H)$ is naturally equivalent to the p-orbit category $\mathcal{O}_p(H)$, with the functor $\mathcal{O}_p(H) \to \mathcal{O}_p(G)$ being given by $H/P \mapsto G/P$. Mathew proves a subgroup decomposition of StMod_{kH} over the orbit category:

5.1. **Theorem** ([Mat16, Corollary 9.16]). *Let $\mathscr{C}$ be a collection of subgroups of H that is closed under intersection and such that every elementary abelian p-subgroup of H is contained in a subgroup in $\mathscr{C}$. There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}_{kH} \xrightarrow{\sim} \lim_{H/K \in \mathcal{O}_{\mathscr{C}}(H)^{\mathrm{op}}} \mathrm{StMod}_{kK}.$$

In light of Theorem 2.1, it is therefore enough to transport the above decomposition, applied to $\mathscr{C} = \mathcal{S}_p(H) \cup \{1\}$, up to StMod_{kG}:

5.2. **Lemma.** *The natural map*

$$\mathrm{StMod}(G/H) \xrightarrow{\sim} \lim_{H/P \in \mathcal{O}_p(H)^{\mathrm{op}}} \mathrm{StMod}(G/P)$$

is an equivalence.

Proof. Let $\mathcal{M}$ denote $\mathrm{Mod}_{\mathrm{StMod}_{kG}}(A_H^G)$; recall from Section 2 that A_H^G is $\mathrm{coind}_H^G(k)$ and that $\mathrm{StMod}(G/H)$ is equal to $\mathcal{M}$ by definition. By [Lur17, Corollary 3.4.1.9], for any $A \in \mathrm{CAlg}(\mathcal{M})$ we have a natural equivalence

$$\mathrm{Mod}_{\mathcal{M}}(A) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{StMod}_{kG}}(A)$$

given by forgetting the A_H^G-module structure. Therefore, we wish to prove that

$$\mathcal{M} \to \lim_{H/P \in \mathcal{O}_p(H)^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{M}}(A_P^G)$$

is an equivalence.

Recall from Theorem 2.1 that coinduction induces a functor

$$\overline{\mathrm{coind}_{\mathrm{H}}^{\mathrm{G}}} : \mathrm{StMod}_{kH} \xrightarrow{\sim} \mathcal{M}$$

that is an equivalence of symmetric monoidal ∞-categories. We obtain a commutative diagram

$$\begin{array}{ccc}
\mathcal{M} & \longrightarrow & \lim_{H/P\in\mathcal{O}_p(H)^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{M}}(A_P^G) \\
\uparrow {\scriptstyle \overline{\mathrm{coind}}_{\mathrm{H}}^{\mathrm{G}}} & & \uparrow \wr \\
 & & \lim_{H/P\in\mathcal{O}_p(H)^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{M}}(\mathrm{coind}_H^G(A_P^H)) \\
 & & \uparrow {\scriptstyle \overline{\mathrm{coind}}_{\mathrm{H}}^{\mathrm{G}}} \\
\mathrm{StMod}_{kH} & \xrightarrow{\sim} & \lim_{H/P\in\mathcal{O}_p(H)^{\mathrm{op}}} \mathrm{Mod}_{\mathrm{StMod}_{kH}}(A_P^H)
\end{array}$$

whose bottom arrow is an equivalence by Mathew's Theorem 5.1. □

We have therefore established:

5.3. **Theorem.** *The right Kan extension of* StMod(−) : $\mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ *along the opposite of the Yoneda embedding* $\mathcal{O}(G) \to \mathcal{S}_G$ *factors through the restriction map*

$$\mathcal{S}_G^{\mathrm{op}} \simeq \mathcal{P}(\mathcal{O}(G))^{\mathrm{op}} \to \mathcal{P}(\mathcal{O}_p(G))^{\mathrm{op}}.$$

In particular, StMod(−) *sends* S*-equivalences in* $\mathcal{S}_G$ *to equivalences of* ∞*-categories.*

5.4. *Remark.* As noted in the introduction, the only part of this argument that was non-formal was checking the descent statement in Lemma 5.2. Therefore, given a collection $\mathscr{F}$ of subgroups of G that is closed under intersections and a functor $F : \mathcal{O}(G)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ that has a subgroup decomposition associated with $\mathscr{F}$, the extended functor $\tilde{F} : \mathcal{S}_G^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ inverts the weak equivalences associated with $\mathscr{F}$.

6. Decompositions of the stable module ∞-category

In this section, we recall Dwyer's construction [Dwy98, Sections 3.4–3.7] of G-spaces that encode candidates for the subgroup, centraliser, and normaliser decompositions. These G-spaces are (homotopy) colimits, so applying StMod(−) gives a limit of ∞-categories that receives a comparison map from StMod_{kG}; the G-space encodes a decomposition precisely when this map is an equivalence. We also have S-homotopy equivalences between these G-spaces, so can use the fact that StMod(−) inverts S-homotopy equivalences of G-spaces to "propagate" Mathew's subgroup decomposition to centraliser and normaliser decompositions.

Let $\mathscr{C}$ be a collection of subgroups of G. Note that we can consider a G-set as a discrete G-space.

Centraliser: Recall from Section 2 that the fusion category $\mathcal{F}_{\mathscr{C}}(G)$ has objects given by the elements of $\mathscr{C}$ and morphisms given by group homomorphisms that are induced by conjugation in G. We have a functor $\alpha : \mathcal{F}_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}_G$ that takes $H \in \mathscr{C}$ to the conjugacy class of the inclusion $H \hookrightarrow G$, which is isomorphic to $G/C_G(H)$ as a G-set. The colimit of α is a G-space that we will denote $\mathrm{E}\mathcal{F}_{\mathscr{C}}(G)$.

Subgroup: We have an inclusion functor $\beta : \mathcal{O}_{\mathscr{C}}(G) \to \mathcal{S}_G$. We let $\mathrm{E}\mathcal{O}_{\mathscr{C}}(G)$ denote the colimit of β.

Normaliser: Recall from Section 2 that the orbit simplex category $\bar{s}S_{\mathscr{C}}(G)$ is the poset of G-conjugacy classes of non-degenerate simplices in the nerve of $\mathscr{C}$, ordered by refinement. We have a functor $\delta : \bar{s}S_{\mathscr{C}}(G)^{\mathrm{op}} \to \mathcal{S}_G$ that takes a G-orbit of simplices $[\sigma]$ to itself, considered as a discrete G-space. This G-set is isomorphic to $G/N_G(\sigma)$, where for $\sigma = (P_0 < \ldots < P_n)$ we define $N_G(\sigma) \coloneqq \bigcap_{0 \le i \le n} N_G(P_i)$. We let $\mathrm{sd}(\mathscr{C})$ denote the colimit of δ.

6.1. *Remark.* Note that $\mathrm{colim}(\delta)$ really is G-equivalent to the subdivision of the nerve of the poset $\mathscr{C}$, justifying our notation. This can be checked directly using the model for $\mathrm{colim}\,\delta$ given by the diagonal of a bisimplicial set that is explained in [BK72, XII 5.2].

6.2. *Remark.* Dwyer calls these three spaces $X^{\alpha}_{\mathscr{C}}$, $X^{\beta}_{\mathscr{C}}$, and $\mathrm{sd}\, X^{\delta}_{\mathscr{C}}$, respectively. For historical reasons, $\mathrm{E}\mathcal{F}_{\mathscr{C}}(G)$ is sometimes called $\mathrm{E}\mathbf{A}_{\mathscr{C}}$ elsewhere in the literature.

Since $\mathrm{StMod}(-)$ sends small colimits of G-spaces to limits of ∞-categories, we get

$$\begin{aligned}\mathrm{StMod}(\mathrm{E}\mathcal{F}_{\mathscr{C}}(G)) &\simeq \lim_{H \in \mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}(\alpha(H)) \\ &\simeq \lim_{H \in \mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}(G/C_G(H)) \\ &\simeq \lim_{H \in \mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}_{kC_G(H)}.\end{aligned}$$

The natural map $\mathrm{E}\mathcal{F}_{\mathscr{C}}(G) \to *$ induces a restriction functor $\mathrm{StMod}_{kG} \to \lim_{\mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}_{kC_G(H)}$, so in this way $\mathrm{E}\mathcal{F}_{\mathscr{C}}(G)$ encodes a candidate for a centraliser decomposition for StMod_{kG}. Similarly, $\mathrm{E}\mathcal{O}_{\mathscr{C}}(G) \to *$ induces a functor

$$\mathrm{StMod}_{kG} \to \lim_{H \in \mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kH}$$

corresponding to a subgroup decomposition and $\mathrm{sd}(\mathscr{C}) \to *$ induces a functor

$$\mathrm{StMod}_{kG} \to \lim_{\sigma \in \bar{s}S_{\mathscr{C}}(G)} \mathrm{StMod}_{kN_G(\sigma)}$$

corresponding to a normaliser decomposition. However, for a general collection $\mathscr{C}$ there is no reason for any of these functors from StMod_{kG} to be an equivalence. We have comparison maps between the G-spaces associated with a collection:

$$\begin{array}{ccccc} & & \mathrm{sd}(\mathscr{C}) & & \\ & & \downarrow \wr & & \\ \mathrm{E}\mathcal{O}_{\mathscr{C}}(G) & \longrightarrow & \mathscr{C} & \longleftarrow & \mathrm{E}\mathcal{F}_{\mathscr{C}}(G) \end{array}$$

Here the map from $\mathrm{sd}(\mathscr{C})$ is the G-equivalence sending a simplex $P_0 < \ldots < P_n$ to P_0; the map from $\mathrm{E}\mathcal{O}_{\mathscr{C}}(G)$ sends a point $x \in G/H$ to its stabiliser G_x; and the map from $\mathrm{E}\mathcal{F}_{\mathscr{C}}(G)$ sends an inclusion $H \hookrightarrow G$ to H.

Let $\mathcal{S}_p(G)$ denote the collection of all non-trivial p-subgroups and $\mathcal{A}_p(G)$ denote the subcollection of non-trivial elementary abelian p-subgroups. Recall that Mathew's Theorem 5.1 shows that for certain collections, including $\mathscr{C} = \mathcal{S}_p(G) \cup \{1\}$ and $\mathscr{C} = \mathcal{A}_p(G) \cup \{1\}$, we obtain a subgroup decomposition. However, to obtain a useful centraliser or normaliser decomposition, we need to remove the trivial subgroup from the collection, otherwise StMod_{kG} itself will appear in the decomposition on the right hand side. We therefore need a minor variation of Theorem 5.1:

6.3. **Lemma.** *Let $\mathscr{C}$ be a collection of subgroups of G that is closed under intersection and such that every elementary abelian p-subgroup of G is contained in a subgroup in $\mathscr{C}$. Let $\mathscr{C}^*$ denote the non-trivial subgroups in $\mathscr{C}$. There is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{StMod}_{kG} \xrightarrow{\sim} \lim_{G/H \in \mathcal{O}_{\mathscr{C}^*}(G)^{\mathrm{op}}} \mathrm{StMod}_{kH}.$$

Proof. Since $\mathrm{StMod}(G/\{1\}) \simeq *$, the diagram of ∞-categories that includes the trivial subgroup is a right Kan extension of the diagram that omits it, so the limits of the two diagrams agree. □

Lemma 6.3 shows we have a subgroup decomposition for $\mathscr{C}^* = \mathcal{S}_p(G)$ and $\mathscr{C}^* = \mathcal{A}_p(G)$. We now transfer this result to other collections and decompositions using the fact that $\mathrm{StMod}(-)$ inverts S-equivalences. We consider the following collections of p-subgroups: let $\mathcal{B}_p(G)$ be the collection of non-trivial p-radical subgroups, *i.e.* non-trivial p-subgroups $P \leq G$ such that P is the maximal normal p-subgroup in $N_G(P)$. Let the collection $\mathcal{I}_p(G)$ consist of all non-trivial subgroups that are intersections of a set of Sylow p-subgroups in G. Finally, let $\mathcal{Z}_p(G)$ be the subcollection of $\mathcal{A}_p(G)$ consisting of those subgroups V such that V is the set of elements in the centre of $C_G(V)$ whose order divides p, *i.e.* such that $V = \Omega_1 O_p Z(C_G(V))$.

6.4. **Theorem.** *Let $\mathscr{C}$ be one of the collections $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{I}_p(G)$, or $\mathcal{Z}_p(G)$. There is a subgroup decomposition*

$$\mathrm{StMod}_{kG} \xrightarrow{\simeq} \lim_{G/P \in \mathcal{O}_{\mathscr{C}}(G)^{\mathrm{op}}} \mathrm{StMod}_{kP}$$

and a normaliser decomposition

$$\mathrm{StMod}_{kG} \xrightarrow{\simeq} \lim_{[\sigma] \in \bar{s}S_{\mathscr{C}}(G)} \mathrm{StMod}_{kN_G(\sigma)}.$$

If $\mathscr{C}$ is $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, or $\mathcal{Z}_p(G)$, then there is additionally a centraliser decomposition

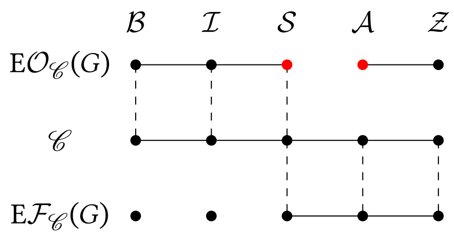
$$\mathrm{StMod}_{kG} \xrightarrow{\simeq} \lim_{P \in \mathcal{F}_{\mathscr{C}}(G)} \mathrm{StMod}_{kC_G(P)}.$$

Proof. We can restate the conclusion of Lemma 6.3 for $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ as saying that

$$\mathrm{E}\mathcal{O}_{\mathcal{S}_p(G)}(G) \to * \quad \text{and} \quad \mathrm{E}\mathcal{O}_{\mathcal{A}_p(G)}(G) \to * \tag{6.5}$$

are both sent to equivalences by $\mathrm{StMod}(-)$.

We now transport this information along S-homotopy equivalences. The following table is taken from [GS06, Theorem 1.1]; a solid line denotes a G-homotopy equivalence, while a dashed line denotes an S-homotopy equivalence. The column labels represent the different collections of subgroups, where for concision we omit p and G from the notation.



The subgroup decompositions arising from the maps in (6.5) correspond to the points $\mathrm{E}\mathcal{O}_{\mathcal{S}}(G)$ and $\mathrm{E}\mathcal{O}_{\mathcal{A}}(G)$, marked in red. The equivalences in the first row show that we have a subgroup decomposition for all of the collections in the table. More interestingly, the S-equivalence $\mathrm{E}\mathcal{O}_{\mathcal{S}}(G) \to \mathcal{S}_p(G)$ induces a normaliser decomposition

$$\begin{array}{ccccc} \lim_{\mathcal{O}_{\mathcal{S}}(G)^{\mathrm{op}}} \mathrm{StMod}(G/P) & \xleftarrow{\sim} & \mathrm{StMod}(\mathcal{S}_p(G)) & \xrightarrow{\sim} & \lim_{\sigma \in \bar{s}S_{\mathcal{S}}(G)} \mathrm{StMod}(G/N_G(\sigma)) \\ & \nwarrow^{\sim} & \uparrow & \nearrow & \\ & & \mathrm{StMod}(G/G) & & \end{array}$$

and hence (via the equivalences in the second row of the table) a normaliser decomposition for all the collections in the table. Finally, the same argument applied to the S-equivalence $\mathrm{E}\mathcal{F}_{\mathcal{S}}(G) \to \mathcal{S}_p(G)$ gives centraliser decompositions for the collections $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, and $\mathcal{Z}_p(G)$. □

6.6. *Remark.* We don't have a condition analogous to Dwyer's "ampleness" for decompositions of the stable module ∞-category. Recall that $\mathscr{C}$ is *ample* if the natural map

$$\mathscr{C}_{\mathrm{h}G} \to (*)_{\mathrm{h}G} \simeq \mathrm{B}G$$

induces an isomorphism on mod p homology, and that there are subgroup, centraliser, and normaliser decompositions associated with $\mathscr{C}$ if and only if $\mathscr{C}$ is ample. This latter statement is due to the maps

$$\begin{array}{ccccc} & & \mathrm{sd}(\mathscr{C}) & & \\ & & \downarrow & & \\ \mathrm{E}\mathcal{O}_{\mathscr{C}}(G) & \longrightarrow & \mathscr{C} & \longleftarrow & \mathrm{E}\mathcal{F}_{\mathscr{C}}(G) \end{array}$$

being hG-homotopy equivalences, *i.e.* G-maps that are homotopy equivalences. Since these maps are not always S-equivalences, there is no analogous condition for the stable module ∞-category. The same phenomenon is discussed in [Dwy98, Remark 3.10] in the context of the behaviour of spectral sequences associated with the decompositions.

Appendix A. All relevant natural transformations are counits

In this section, we prove the assertion made in Remark 4.3 that the natural transformation in each step of the outline of Theorem 4.8 is given by the counit of some adjunction. The only step for which this is not obvious is the following:

A.1. **Lemma.** *Under the equivalence*

$$(i^*)^*(i_!)^*(y_B)_! \xrightarrow{\sim} (y_B)_! j_! j^*$$

given by Lemmas 4.5 and 4.6, the natural transformation

$$(i^*)^*(i_!)^*(y_B)_! \to (y_B)_! \tag{A.2}$$

induced by the counit of the $(i^*)^* \dashv (i_!)^*$ *adjunction corresponds to the natural transformation*

$$(y_B)_! j_! j^* \to (y_B)_! \tag{A.3}$$

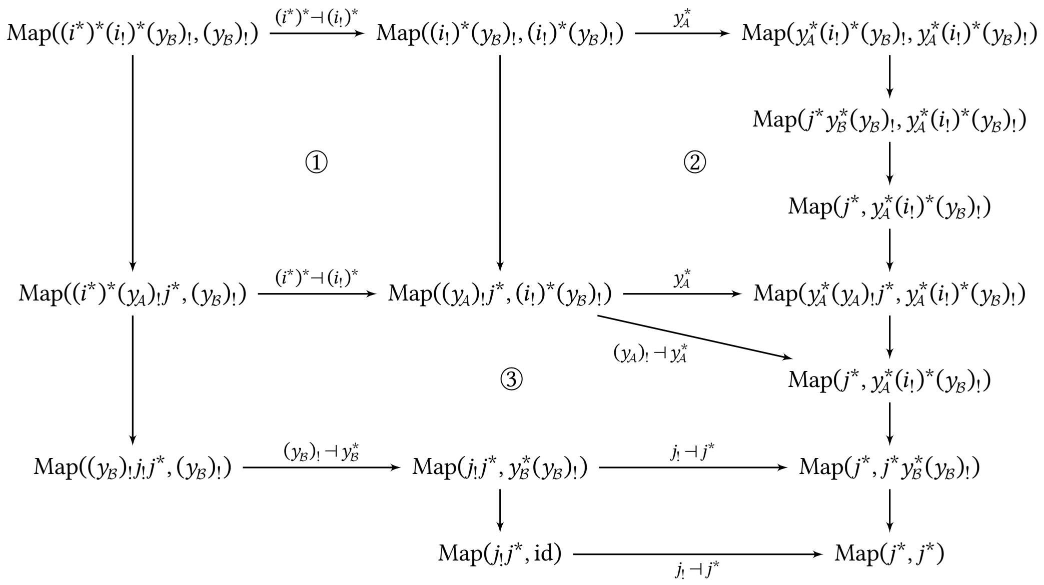
induced by the counit of the $j_! \dashv j^*$ *adjunction.*

Proof. This amounts to a large diagram chase; the diagram is reproduced below. We first explain why the diagram proves the lemma, then explain why the diagram commutes. Every map in the diagram is an equivalence. For functors $F, F' \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we denote the mapping space of natural transformations from F to F' by $\mathrm{Map}(F, F')$.

The natural transformation (A.2) is an element of the top-left mapping space in the diagram, and the two vertical maps on the left-hand edge are induced by the equivalences of Lemmas 4.5 and 4.6. We therefore aim to show that (A.2) is sent to (A.3) in the bottom-left corner of the region labelled ③.

Since (A.2) is induced by the counit of the $(i^*)^* \dashv (i_!)^*$ adjunction, it is sent to the identity natural transformation in the top-right corner of the square labelled ②. The maps along the right-hand edge of the diagram are either induced by cancelling inverse equivalences or by applying the identity $y_A^*(i_!)^* \simeq j^* y_B^*$ of Diagram 4.7. One can check that these maps send the identity natural transformation in the top-right corner of ② to the identity natural transformation in the bottom-right corner of the diagram, which in turn is sent to (A.3) in the bottom-left corner of the region labelled ③.

Therefore, it remains to establish that the diagram commutes. The square labelled ② commutes by definition of its left-hand edge; see Lemma 4.5. The region labelled ③ also commutes by definition of its left-hand edge; see Lemma 4.6. All the other regions are easily seen to commute. □



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